Group-type partial actions of groupoids and a Galois correspondence

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The usual notion of Galois extension over fields was extended for commutative rings by M. Auslander and O. Goldman in [1]. Some years later, the Galois theory over commutative rings was developed by S. U. Chase, D. K. Harrison and A. Rosenberg in [6]. They presented several equivalent conditions for the definition of Galois extension. Among the main results, they proved a Galois correspondence in the context of commutative rings. Precisely, if $R \subset S$ is a Galois extension of commutative rings with Galois group G, then there exists a bijective association between the set of subgroups of G and the set of R-subalgebras of S which are G-strong and R-separable.

In the 1990's, R. Exel introduced the notion of partial actions of a group in the theory of operator algebras, see for instance [9] and [10]. The same notion in an algebraic context was considered in [7]. Particularly, it was defined partial actions of groups on rings which is the key to develop a partial Galois theory. So, the Galois theory for partial actions of groups on rings was presented two years later in [8] generalizing the results of [6].

On the other hand, in the context of category theory, a groupoid is a small category in which every morphism has inverse. However, a groupoid can be seen as a natural generalization of a group. In fact, a groupoid is a set G equipped with a set of identities $\mathsf{G}_0 \subset \mathsf{G}$ and a binary operation defined partially which is associative and, for each $g \in \mathsf{G}$, there exist $g^{-1} \in \mathsf{G}$ such that $g^{-1}g = s(g) \in \mathsf{G}_0$ and $gg^{-1} = t(g) \in \mathsf{G}_0$. If G_0 has a unique element then G is a group. This algebraic version of groupoids motivated the authors of [2] to consider partial actions of groupoids on rings. In particular, it was defined in [2] the notion of Galois extension for partial actions of groupoids. A version of the Galois correspondence for global actions of groupoids on commutative rings was given in [11].

An special class of partial actions of connected groupoids was studied in [4]. This class was named *group-type* partial groupoid actions and this name is due to the fact that the partial skew groupoid ring associated can be realized as a partial skew group ring; see details in Theorem 4.4 of [4]. It is easy to construct examples of group-type partial actions of groupoids using the formulas given in (4) and (5) of [3]. In particular, every global groupoid action is a group-type partial action.

The main contribution of this talk is to show a Galois correspondence for group-type partial actions of groupoids. This correspondence is submitted in a recent paper joint with D. Bagio and A. Sant'Ana (cf. [5]). Precisely, let $\alpha = (S_g, \alpha_g)_{g\in \mathsf{G}}$ be a unital group-type partial action of a connected finite groupoid G on a comutative ring $S = \oplus_{y\in \mathsf{G}_0} S_y$. For each subgroupoid H of G, we consider $\alpha_{\mathsf{H}} = (S_h, \alpha_h)_{h\in \mathsf{H}}$ the partial action of H on $S_{\mathsf{H}} = \oplus_{y\in \mathsf{H}_0} S_y$. Denote by $S^{\alpha_{\mathsf{H}}}$ the subring of invariant elements. On the other hand, G_T denotes the set of elements of G that fix T, where T is a subring of S. Consider the set $\boldsymbol{wSubg}(\mathsf{G})$ whose elements are wide subgroupoids H of G such that α_{H} is group-type. Also, let $\mathbf{B}(S)$ be the set of all subrings T of S which are $S^{\alpha_{\mathsf{G}}}$-separable, α-strong and such that $\mathsf{G}_T = \mathsf{H}$, for some $\mathsf{H} \in \boldsymbol{wSubg}(\mathsf{G})$. With this notation, we have the following Galois correspondence.

Theorem. (Galois Correspondence) *Let S be an α_{G}-partial Galois extension of $S^{\alpha_{\mathsf{G}}}$. There exists a bijective correspondence between $\boldsymbol{wSubg}(\mathsf{G})$ and $\boldsymbol{B}(S)$ given by $\mathsf{H} \mapsto S^{\alpha_{\mathsf{H}}}$ whose inverse is given by $T \mapsto \mathsf{G}_T$.*

The Galois correspondence for not-necessarily connected groupoids follows from the connected case. The previous theorem recover the Galois correspondence for global groupoid actions given in Theorem 4.6 (i) of [11].

References

[1] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. 97 (1960), 367–408.

[2] D. Bagio, A. Paques, *Partial groupoid actions: globalization, Morita theory and Galois theory*, Comm. Algebra 40 (10) (2012), 3658–3678.

[3] D. Bagio, A. Paques; H. Pinedo *Restriction and extension of partial actions*, J. Pure Appl. Algebra 224 (10) (2020), p.106391.

[4] D. Bagio, A. Paques and H. Pinedo, *On partial skew groupoids rings*, Int. J. Algebra Comput. 31 (1) (2021), 1–17

[5] D. Bagio, A. Sant'Ana and T. Tamusiunas, *Galois correspondence for groupoid partial actions*, preprint, arXiv:2108.01180 (2021).

[6] S. U. Chase, D. K. Harrison, A. Rosenberg, *Galois theory and Galois cohomology of comutative rings*, Mem. Amer. Math. Soc. 52 (1968), 1-19.

[7] M. Dokuchaev and R. Exel, *Associativity of crossed products by partial actions, enveloping actions and partial representations*, Trans. Amer. Math. Soc. 357 (5) (2005), 1931–1952.

[8] M. Dokuchaev, M. Ferrero and A. Paques, *Partial actions and Galois theory.* J. Pure Appl. Algebra 208 (2007), 77–87.

[9] R. Exel, *Twisted partial actions: a classification of regular C^*-algebraic bundles.* Proc. London Math. Soc. 74 (1997), 417–443.

[10] R. Exel, *Partial actions of groups and actions of semigroups.* Proc. Amer. Math. Soc. 126 (12) (1998), 3481–3494.

[11] A. Paques; T. Tamusiunas, *The Galois correspondence theorem for groupoid actions*, J. Algebra 509 (2018), 105–123.