# Representations of the restricted enveloping algebra $\mathfrak{u}(\mathfrak{m})$ in characteristic 2

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* This is a work in progress with N. Andruskiewitsch, S. D. Flora and D. Flores.

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- $D(H)$ as an abelian extension:

$$\mathbf{T} \xhookrightarrow{\iota} D(H) \xrightarrow{\pi} \mathfrak{u}(\mathfrak{m})$$

- $\mathfrak{u}(\mathfrak{m})$ is of tame representation type.
- Indecomposable representations of $\mathfrak{u}(\mathfrak{m})$.

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## Restricted Jordan plane

It is the algebra $\mathscr{B}$ of dimension $2^4$ presented by generators $x_1, x_2$ with defining relations

$$x_1^2 = 0, \qquad x_2^2 x_1 = x_1 x_2^2 + x_1 x_2 x_1, \tag{1}$$

$$x_2^4 = 0, \qquad x_1 x_2 x_1 x_2 = x_2 x_1 x_2 x_1. \tag{2}$$

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## Bosonization

Let $\Gamma = \langle g \rangle$ be the cyclic group of order 2, written multiplicatively. The bosonization $H := \mathscr{B} \# \Bbbk\Gamma$ is a pointed Hopf algebra of dimension $2^5$ generated by $x_1, x_2, g$ with satisfies the previous relations and

$$g x_1 = x_1 g, \qquad g x_2 = x_2 g + x_1 g, \qquad g^2 = 1. \tag{3}$$

The coproduct of $H$ is given by

$$\Delta(g) = g \otimes g, \qquad \Delta(x_i) = x_i \otimes 1 + g \otimes x_i, \quad i \in \mathbb{I}_2.$$

## Drinfeld double of $H$

The Drinfeld double of $H$ is $D(H) = H \otimes H^{*\,\mathrm{op}}$ as coalgebra. As algebra, $D(H)$ is generated by $x_1, x_2, g, w_1, w_2, \gamma$ with relations (1),(2), (3) and

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w_2^4 &= 0, & w_1 w_2 w_1 w_2 &= w_2 w_1 w_2 w_1,\\
\gamma^2 &= \gamma, & w_i\gamma &= \gamma w_i + w_i,\\
w_1 x_1 &= x_1 w_1, & w_1 x_2 &= x_2 w_1 + 1 + g,\\
w_1 g &= g w_1, & w_2 x_1 &= x_1 (w_1 + w_2) + 1 + g,\\
w_2 g &= g(w_1 + w_2), & \gamma x_i &= x_i \gamma + x_i,\\
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w_1 x_1 &= x_1 w_1, & w_1 x_2 &= x_2 w_1 + 1 + g,\\
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\end{aligned}$$

We have $\dim D(H) = 2^{10}$.

Fix the following elements in $D(H)$:

$$x_{21} = x_1 x_2 + x_2 x_1, \qquad w_{21} = w_1 w_2 + w_2 w_1.$$

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## Central Hopf subalgebra

The subalgebra $\mathbf{T}$ of $D(H)$ generated by $x_1$, $x_{21}$, $w_1$, $w_{21}$ and $g$ is a normal local commutative Hopf subalgebra with defining relations

$$x_1^2 = 0, \quad x_{21}^2 = 0, \quad w_1^2 = 0, \quad w_{21}^2 = 0, \quad g^2 = 1.$$

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Also $\dim \mathbf{T} = 2^5$.

Hence

$$\mathbf{T} \overset{\iota}{\hookrightarrow} D(H) \overset{\pi}{\twoheadrightarrow} D(H)/D(H)\mathbf{T}^+$$

is an exact sequence of Hopf algebras.

We fix the following elements of $D(H)/D(H)\mathbf{T}^+$:

$$a = \overline{x}_2, \qquad b = \overline{w}_2, \qquad c = \overline{\gamma}.$$

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### Hopf algebra quotient

The algebra $D(H)/D(H)\mathbf{T}^+$ is generated by $a, b, c$ and satisfies the relations

$$ab + ba = c, \qquad ac + ca = a, \qquad bc + cb = b,$$
$$a^4 = b^4 = 0, \qquad c^2 + c = 0.$$

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The Hopf algebra $D(H)/D(H)\mathbf{T}^+$ is a well-known algebra in modular Lie theory.

Denote by $\mathfrak{s}$ the unique, up to isomorphism, simple Lie algebra of dimension 3, that is, $\mathfrak{s}$ has a basis $\{e, f, h\}$ and bracket

$$[e, f] = h, \qquad [e, h] = e, \qquad [f, h] = f.$$

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## The algebra $\mathfrak{u}(\mathfrak{m})$

The restricted enveloping algebra $\mathfrak{u}(\mathfrak{m})$ of $\mathfrak{m}$ is isomorphic to $D(H)/D(H)\mathbf{T}^+$ via

$$e \mapsto a, \qquad f \mapsto b, \qquad h \mapsto c.$$

and we have an exact sequence of Hopf algebras

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$$\mathbf{T} \overset{\iota}{\hookrightarrow} D(H) \overset{\pi}{\twoheadrightarrow} \mathfrak{u}(\mathfrak{m})$$

For this reason we are interested in the representations of $\mathfrak{u}(\mathfrak{m})$.

Let $V_0$, respectively $V_1$, denote the one-dimensional $\mathfrak{u}(\mathfrak{m})$-module, respectively the three dimensional $\mathfrak{u}(\mathfrak{m})$-module $\mathfrak{s}$ with the adjoint representation ad.

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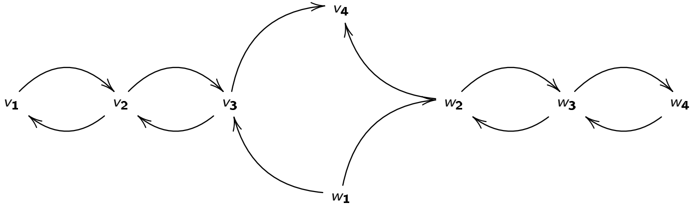
Thus $V_1$ in the basis $\{v_1, v_2, v_3\} := \{b, c, a\}$ of $\mathfrak{s}$ is given by ad $a = \mathtt{A}$, ad $b = \mathtt{B}$, ad $c = \mathtt{C}$, where

$$\mathtt{A} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathtt{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathtt{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Theorem.**
The simple modules of $\mathfrak{u}(\mathfrak{m})$ are $V_0$ and $V_1$.

Consider the following 8-dimensional $\mathfrak{u}(\mathfrak{m})$-module $M$




where the arrows oriented from left to right indicate the action of $a$ while the arrows from right to left are the action of $b$.

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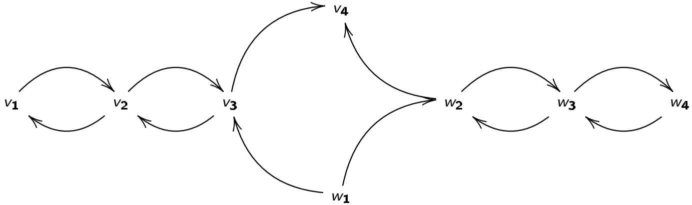


where the arrows oriented from left to right indicate the action of $a$ while the arrows from right to left are the action of $b$.

**Theorem.**

- The projective cover of the simple module $V_0$ is $P(V_0) = M$.
- $P(V_0) \simeq \mathfrak{u}(\mathfrak{m})e_0$, where $e_0 = (1 + ab + a^2b^2)(1 + c)$ is a primitive idempotent of $\mathfrak{u}(\mathfrak{m})$.

Consider the following 8-dimensional $\mathfrak{u}(\mathfrak{m})$-module $N$

$$
\begin{tikzcd}
 & v_2 \arrow[r, bend left] & v_3 \arrow[l, bend left] \arrow[r, bend left] & v_4 \arrow[l, bend left] & \\
v_1 \arrow[ur, bend left] & & & & w_4 . \arrow[ul, bend right] \\
 & w_1 \arrow[ul, bend left] \arrow[r, bend left] & w_2 \arrow[l, bend left] \arrow[ul] \arrow[r, bend left] & w_3 \arrow[l, bend left] \arrow[ul] \arrow[ur, bend left] &
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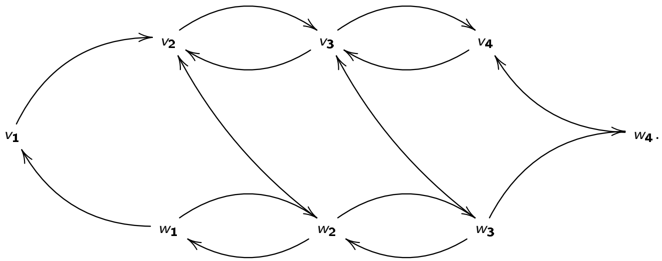


## Theorem.

- The projective cover of the simple module $V_1$ is $P(V_1) = N$.
- $P(V_1) \simeq \mathfrak{u}(\mathfrak{m})e_1$, where $e_1 = (1 + a^2b^2)c$ is a primitive idempotent of $\mathfrak{u}(\mathfrak{m})$.

Let $e = e_0 + e_1$ where $e_0$ and $e_1$ are the primitive idempotents in $\mathfrak{u}(\mathfrak{m})$ generating the projective covers $P(V_0)$ and $P(V_1)$. Then the basic algebra associated to $\mathfrak{u}(\mathfrak{m})$ is

$$\mathfrak{u}(\mathfrak{m})^{\mathfrak{b}} = e\mathfrak{u}(\mathfrak{m})e.$$

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The basic algebra $\mathfrak{u}(\mathfrak{m})^{\mathrm{b}}$ has a basis

$$\{e_0, e_1, ae_0, a^3e_1, b^3e_0, be_1, a^3b^3e_0, abe_1\}.$$

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The ordinary quiver of $\mathfrak{u}(\mathfrak{m})^{\mathrm{b}}$, denoted by $Q := Q_{\mathfrak{u}(\mathfrak{m})^{\mathrm{b}}}$, is

$$\begin{tikzcd} 1 \arrow[r, bend left=50, "\alpha_1"] \arrow[r, bend left=20, "\alpha_2"] & 2. \arrow[l, bend left=20, "\beta_1"] \arrow[l, bend left=50, "\beta_2"] \end{tikzcd}$$

## Theorem.

We have that $\mathfrak{u}(\mathfrak{m})^{\mathfrak{b}} \simeq \Bbbk Q/I$, where

$$I = \langle \alpha_1\beta_1, \alpha_2\beta_2, \beta_1\alpha_1, \beta_2\alpha_2, \alpha_1\beta_2 + \alpha_2\beta_1, \beta_1\alpha_2 + \beta_2\alpha_1 \rangle$$

is the kernel of the algebra epimorphism $\varphi : \Bbbk Q \to \mathfrak{u}(\mathfrak{m})^{\mathfrak{b}}$ defined by

$$\varphi(\alpha_1) = a^3 e_1, \quad \varphi(\alpha_2) = be_1, \quad \varphi(\beta_1) = ae_0, \quad \varphi(\beta_2) = b^3 e_0,$$

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Since $\mathfrak{u}(\mathfrak{m})$ is Morita equivalent to $\mathfrak{u}(\mathfrak{m})^{\mathrm{b}}$ it follows that $\mathfrak{u}(\mathfrak{m})$ is tame representation type.

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The words $a = \alpha_1 \alpha_1^{-1} \alpha_2$ and $b = \alpha_1 \beta_2$ are not string. In fact, $a$ is not a string because $\alpha_1 \alpha_1^{-1}$ is a "piece" of $a$ and $b$ is not a string because $\alpha_1 \beta_2$ is a monomial of the binomial relation $\alpha_1 \beta_2 + \alpha_2 \beta_1$.

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The words $s = \alpha_1\alpha_2^{-1}\alpha_1$ and $t = \beta_1^{-1}\beta_2$ are examples of strings.

## String

Consider the words $s_1 = \alpha_1\alpha_2^{-1}$, $s_2 = \alpha_1^{-1}\alpha_2$, $s_3 = \beta_1\beta_2^{-1}$ and $s_4 = \beta_1^{-1}\beta_2$ and $r$ an integer. The families of string in $Q$ are the following:

$$
\begin{aligned}
&w_1(r) = s_1^r, && w_2(r) = s_2^r, && r \geq 1,\\
&w_3(r) = s_3^r, && w_4(r) = s_4^r, && r \geq 1,\\
&w_5(r) = s_1^r\alpha_1, && w_6(r) = (s_1^{-1})^r\alpha_2, && r \geq 0\\
&w_7(r) = s_3^r\beta_1, && w_8(r) = (s_3^{-1})^r\beta_2, && r \geq 0.
\end{aligned}
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$$\begin{array}{lll} w_1(r) = s_1^r, & w_2(r) = s_2^r, & r \geq 1, \\ w_3(r) = s_3^r, & w_4(r) = s_4^r, & r \geq 1, \\ w_5(r) = s_1^r\alpha_1, & w_6(r) = (s_1^{-1})^r\alpha_2, & r \geq 0 \\ w_7(r) = s_3^r\beta_1, & w_8(r) = (s_3^{-1})^r\beta_2, & r \geq 0. \end{array}$$

Similarly, we have the notion of band. For our case, there are 2 families of band in $Q$.

In order to illustrate how to associate an indecomposable module to a string, we consider the string $w_1(1) = s_1 = \alpha_1\alpha_2^{-1}$:

$$1 \xrightarrow{\alpha_1} 2 \xleftarrow{\alpha_2} 1$$

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The right $\Bbbk Q/I$-module $U(w_1(1)) := \Bbbk\{u_1, u_2, u_3\}$ (a vector for each vertex) is given by:

$$\begin{array}{llll} u_1 \cdot \epsilon_1 = u_1, & u_1 \cdot \epsilon_2 = 0, & u_1 \cdot \alpha_1 = u_2, & u_1 \cdot \alpha_2 = 0, \\ u_2 \cdot \epsilon_1 = 0, & u_2 \cdot \epsilon_2 = u_2, & u_2 \cdot \alpha_1 = 0, & u_2 \cdot \alpha_2 = 0, \\ u_3 \cdot \epsilon_1 = u_3, & u_3 \cdot \epsilon_2 = 0, & u_3 \cdot \alpha_1 = 0, & u_3 \cdot \alpha_2 = u_2, \end{array}$$

The algebra isomorphism $\mathfrak{u}(\mathfrak{m})^{\mathfrak{b}} \simeq \Bbbk Q/I$ and an anti-isomorphism of Hopf algebras $\mathfrak{u}(\mathfrak{m})^{\mathfrak{b}} \to \mathfrak{u}(\mathfrak{m})^{\mathfrak{b}}$ induce on $U(w_1(1))$ a left $\mathfrak{u}(\mathfrak{m})^{\mathfrak{b}}$-module structure via

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e_1 \cdot u_1 = u_1, & e_2 \cdot u_1 = 0, & ae_1 \cdot u_1 = u_2, & b^3 e_1 \cdot eu_1 = 0, \\
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\end{array}
$$

The other elements of the basis of $\mathfrak{u}(\mathfrak{m})^{\mathfrak{b}}$ act trivially.

The functors

$$\begin{aligned} \mathrm{Ind}_e &:= {}_{\mathfrak{u}(\mathfrak{m})^\flat}\mathcal{M} \to {}_{\mathfrak{u}(\mathfrak{m})}\mathcal{M}, & \mathrm{Ind}_e(N) &= \mathfrak{u}(\mathfrak{m})e \otimes_{\mathfrak{u}(\mathfrak{m})^\flat} N, \\ \mathrm{Res}_e &:= {}_{\mathfrak{u}(\mathfrak{m})}\mathcal{M} \to {}_{\mathfrak{u}(\mathfrak{m})^\flat}\mathcal{M}, & \mathrm{Res}_e(M) &= eM \end{aligned}$$

are inverse equivalences of categories.

The functors

$$\begin{aligned} \mathrm{Ind}_e : &= {}_{\mathfrak{u}(\mathfrak{m})^{\mathfrak{b}}}\mathcal{M} \to {}_{\mathfrak{u}(\mathfrak{m})}\mathcal{M}, &\qquad \mathrm{Ind}_e(N) &= \mathfrak{u}(\mathfrak{m})e \otimes_{\mathfrak{u}(\mathfrak{m})^{\mathfrak{b}}} N, \\ \mathrm{Res}_e : &= {}_{\mathfrak{u}(\mathfrak{m})}\mathcal{M} \to {}_{\mathfrak{u}(\mathfrak{m})^{\mathfrak{b}}}\mathcal{M}, &\qquad \mathrm{Res}_e(M) &= eM \end{aligned}$$

are inverse equivalences of categories.

Thus $\mathrm{Ind}_e\left(U(w_1(1))\right)$ is the following 5-dimensional left $\mathfrak{u}(\mathfrak{m})$-module

$$\begin{tikzcd} e_1 \otimes u_1 \arrow[r, bend left] & e_2 \otimes u_2 \arrow[r, bend left] & ae_2 \otimes u_2 \arrow[r, bend left] \arrow[l, bend left] & a^2 e_2 \otimes u_2 \arrow[l, bend left] & e_1 \otimes u_3 \arrow[l, bend left] \end{tikzcd},$$

where the arrows oriented from left to right indicate the action of $a$ while the arrows from right to left are the action of $b$.

## Category of finite-dimensional indecomposable left $\mathfrak{u}(\mathfrak{m})$-modules

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## Next step

Determine the fusion rules. Precisely, for all finite-dimensional indecomposable left $\mathfrak{u}(\mathfrak{m})$-modules $U$ and $V$, calculate the decomposition of $U \otimes_{\mathbb{k}} V$ in direct sum of indecomposable modules.

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Thank you!