Leavitt Path Algebras as Partial Skew Group Rings

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Abstract

We realize Leavitt path algebras as partial skew group rings, making a modification to the proposed construction in [3]. Also, we will mention some theorems for which it is useful to use this realization.

1. Introduction

Leavitt path algebras over a field were initially introduced by Gene Abrams and Aranda Pino (see [1]) as generalizations of Cuntz-Kriger Algebras. On the other hand, partial skew group rings were introduced by Ruy Exel and Mikhailo Dokuchaev as algebraic analogs of C^* partial cross products (see [2]). In this poster we are going to show the isomorphism between the Leavitt path algebras and a particular Partial Skew Group Ring and as a direct consequence, we will obtain a new grading on Leavitt path algebras. Furthermore, this will make the Lavitt path algebra theory to benefit from the Partial Skew Group Ring theory.

2. The partial action

Let $E = (E^1, E^0, r, s)$ a graph without isolated vertices (i.e, for all $v \in E^0$ we have to $r^{-1}(v) \neq \emptyset$ or $s^{-1}(v) \neq \emptyset$) and $\mathbb{F}$ the free group generated by E^1. We denote the set of all finite paths in E by W, and the set of all infinite paths in E by W^∞ and let

$$X = \{\xi \in W : r(\xi) \textit{ is a sink}\} \cup \{v \in E^0 : v \textit{ is a sink }\} \cup W^\infty.$$

We are going to define the set X_c for $c \in \mathbb{F}$ as follows:

- $X_0 := X$, where 0 is the neutral element of $\mathbb{F}$;
- $X_{b^{-1}} := \{\xi \in X \mid s(\xi) = r(b)\}$, for all $b \in W$;
- $X_a := \{\xi \in X \mid \xi_1\xi_2 \cdots \xi_{|a|} = a\}$ for all $a \in W$;
- $X_{ab^{-1}} := \{\xi \in X \mid \xi_1\xi_2 \cdots \xi_{|a|} = a\} = X_a$, for $ab^{-1} \in \mathbb{F}$ with $a, b \in W$, $r(a) = r(b)$ and ab^{-1} in its reduce form;
- $X_g := \emptyset$, for all other $g \in \mathbb{F}$.

Remark 1
Note that $r(b) \in X_{b^{-1}}$ if and only if $r(b)$ is a sink, and moreover, if $r(b)$ is a sink, then $X_{b^{-1}} = \{r(b)\}$ and $X_b = \{b\}$.

We obtain a partial action in the level set $\alpha = (X_g, \alpha_g)_{g \in \mathbb{F}}$ we can bring this partial action in the algebra level by taking $F(X)$ be the $K-$algebra of the functions from X to K with pointwise multiplication, for each $g \in \mathbb{F}$, with $X_g \neq \emptyset$ let $F(X_g)$ be the $K-$algebra of functions from X_g to K and the isomorphism $\alpha_g : F(X_{g^{-1}}) \to F(X_g)$ by $\alpha_g(f)) = f \circ \theta_g$ for all $g \in \mathbb{F}$. Thus the family $\alpha = (\alpha_g, F(X_g))_{g \in \mathbb{F}}$ is a partial action at the algebra level. To obtain the Leavitt path algebra we need to make our algebra "smaller".

We can now define the partial action which induces the partial skew group ring that is isomorphic to the Leavitt path algebra. Let v an infinite emitter, consider the formal series $\sum_{i=1}^{\infty} 1_{(e_i)_v}$ where e_v denotes an edge such that $s(e) = v$, let

$$D(X) = span \left\{ \{1_g \mid g \in \mathbb{F} \setminus \{0\}\} \cup \left\{ \sum_{i=1}^{\infty} 1_{(e_i)_v} : v \textit{ is an infinite emitter} \right\} \right\}$$

(where span means the $K-$linear span) and, for each $g \in \mathbb{F} \setminus \{0\}$, let $D_g \subseteq F(X_g)$ be defined as $1_g D(X)$; that is,

$$D_p = span\{1_p 1_q \mid q \in \mathbb{F}\}.$$

The operations in $D(X)$ are defined as follows:

- $\sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} (a_i + b_i)$;
- $k \cdot \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} k \cdot a_i$ for all $k \in K$;
- $\left(\sum_{i=1}^{\infty} a_i\right) \left(\sum_{i=1}^{\infty} b_i\right) = \left(\sum_{i=1}^{\infty} c_i\right)$ where $c_i = \sum_{i=m+n} a_m a_n$.

Remark 2
we can consider the linear combinations of the set $\{1_g \mid g \in \mathbb{F} \setminus \{0\}\}$ as formal series where $1_g = 0$ for all but finitely many $g \in \mathbb{F}$. Thus, the above operations are defined for all $D(X)$.

Consider for each $p \in \mathbb{F}$ the restriction of $\alpha_p : D_{p^{-1}} \to D_p$. Notice that α_p is an isomorphism of $K-$algebras. Thus, $\alpha = (\alpha_g, D_g)_{g \in \mathbb{F}}$ is a partial action. Let $D(X) \rtimes_\alpha \mathbb{F}$ be the partial skew group ring associated to it.

3. The Leavitt Path Algebra and Skew Group Rings

Proposition 3
There exists a $K-$homomorphism φ, from $L_K(E)$ on $D(X) \rtimes \mathbb{F}$. Where E is a graph without isolated vertices.

Proof. Consider the sets $\{1_e \mid e \in E^1\}$ and $\{1_{e^{-1}} \mid e \in E^1\}$. Now, we want to find how to replace the characteristic functions 1_v with $v \in E^0$. We have the following three cases:

I. If v is a sink: Since the graph has no isolated vertices then exist $e \in E^1$ such that $r(e) = v$, by the axiom of choice we can choose $e_v \in \{f \in E^1 \mid r(f) = v\}$ thus we replace the 1_v function by $1_{e_v^{-1}}$ (Notice that $1_v = 1_{e_v^{-1}}$ by Remark 1).

II. If v is a regular vertex: By the second Cuntz-Krieger property we replace 1_v by the finite sum $\sum_{i=1}^{n} 1_{e_{i_v}}$ where $\{e \in E^1 \mid s(e) = v\} = \{(e_1)_v, \cdots, (e_n)_v\}$.

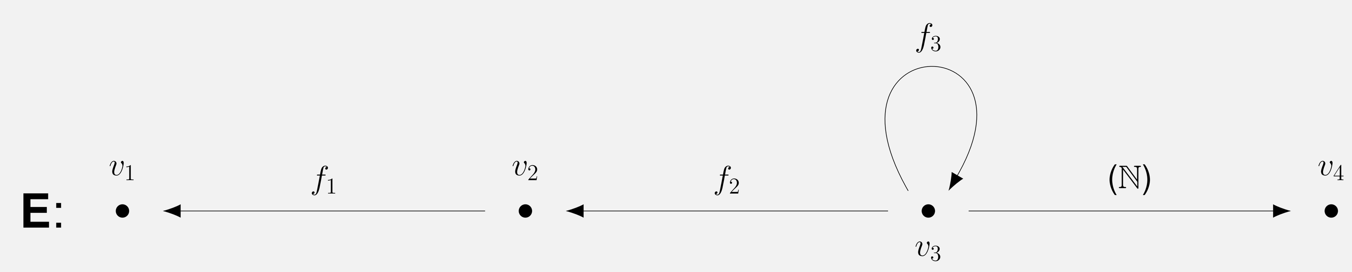
III. If v is an infinite emitter: We replace the function 1_v by the formal series $\sum_{i=1}^{\infty} 1_{(e_i)_v}$, this because the set $\{e_i \in E^1 \mid s(e_i) = v\} = X_v = \bigcup_{i=1}^{\infty} X_{e_i}$.

We can prove by following [3, Proposition 3.2] that the sets $\{1_e \mid e \in E^1\}$, $\{1_{e^{-1}} \mid e \in E^1\}$ and $\{1_{e_v^{-1}} \delta_0 \mid v \textit{ is a sink}\} \cup \left\{ \sum_{i=1}^{n} 1_{e_{i_v}} \mid v \textit{ is a regular vertex} \right\} \cup \left\{ \sum_{i=1}^{\infty} 1_{(e_i)_v} \mid v \textit{ is an infinite emitter} \right\}$ satisfy the relations that define the Leavitt path algebra and then we use the universal property of $L_K(E)$ to obtain the desired homomorphism. □

Theorem 4
The homomorphism $L_K(E) \to D(X) \rtimes \mathbb{F}$ introduced in the previous proposition is a $K-$isomorphism.

Proof. Proof is similar to [3, Theorem 3.3]. □

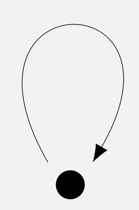
Example 5
If we consider the graph:



Then $L_K(E) \cong D(X) \rtimes \mathbb{F}$.

4. Applications

Theorem 6
Let E a graph without sources. $D(X) \rtimes \mathbb{F} \cong L_K(E)$ is strongly graded if and only if E is a loop.



Theorem 7
Let K be a field and let E be a directed graph. Consider $L_K(E)$, the Leavitt path $K-$algebra of E with coefficients in K. Then E is finite and acyclic if and only if $L_K(E)$ is artinian.

Proof. See [4, Theorem 5.2]. □

References

[1] G. Abrams and G. A. Pino. The leavitt path algebra of a graph. *Journal of Algebra*, 293(2):319–334, 2005.

[2] M. Dokuchaev and R. Exel. Associativity of crossed products by partial actions, enveloping actions and partial representations. *Transactions of the American Mathematical Society*, 357(5):1931–1952, 2005.

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[4] P. Nystedt, J. Öinert, and H. Pinedo. Artinian and noetherian partial skew groupoid rings. *Journal of Algebra*, 503:433–452, 2018.