Simplicity of Nekrashevych algebras of contracting self-similar groups

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Cimpa School — From Dynamics to Algebra and Representation Theory and Back

Outline

Rooted trees

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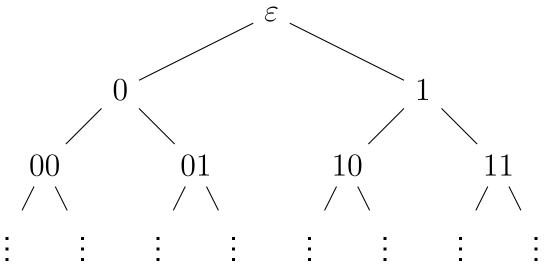

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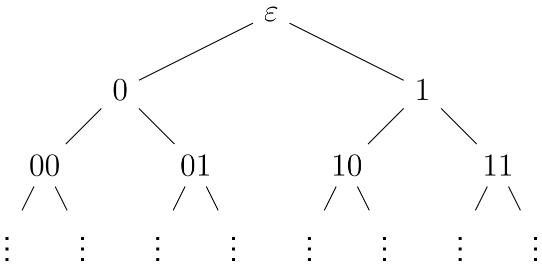

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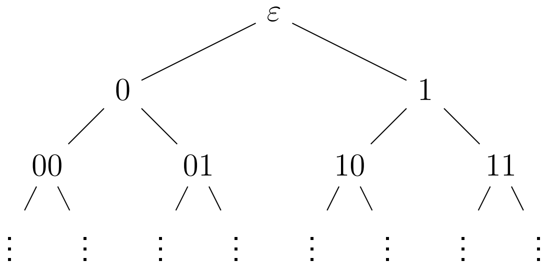

- The boundary ∂T_X is homeomorphic to X^ω.
- So $\mathrm{Aut}(T_X)$ acts on X^ω.

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- Note that $g|_{x_1 \cdots x_n} = ((g|_{x_1})|_{x_2} \cdots)|_{x_n}$.

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- Bartholdi and Nekrashevych used self-similar groups called iterated monodromy groups to solve Hubbard's twisted rabbit problem in complex dynamics.
- Any self-similar subgroup of $\mathrm{Aut}(T_X)$ acts on X^ω.

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- Automaton groups always have decidable word problem.

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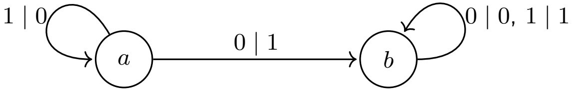

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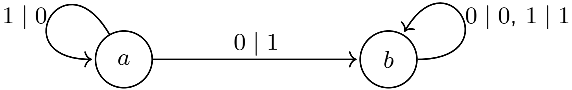

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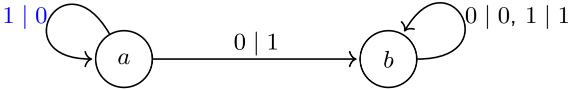

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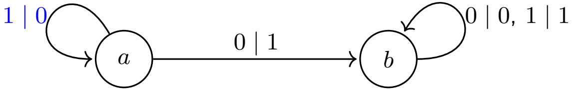

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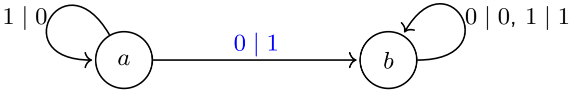

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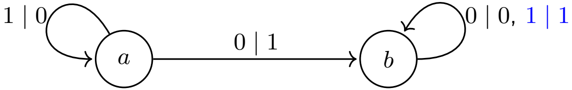

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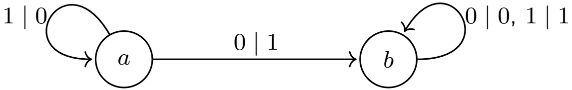

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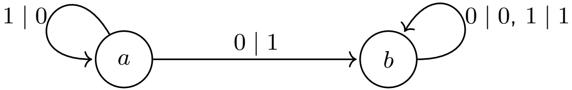

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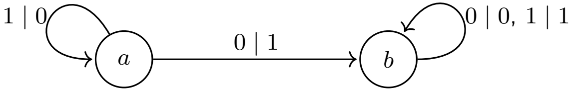

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- $\langle a, b\rangle \cong \mathbb{Z}$.

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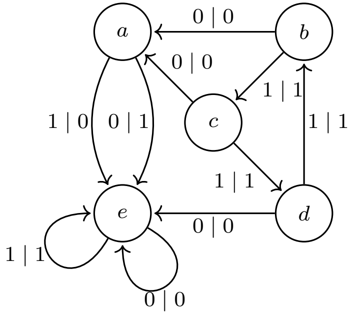

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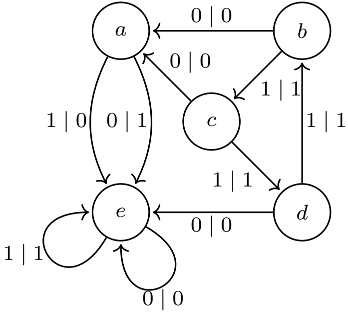

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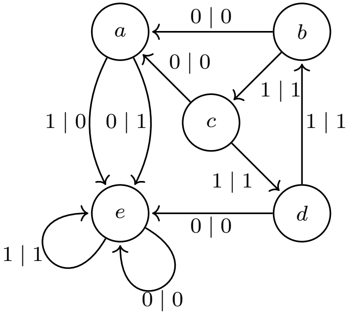

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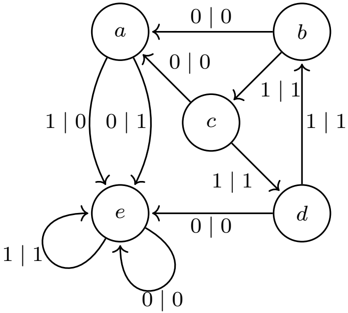

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- It is a finitely generated, just infinite 2-group of intermediate growth.
- It was the first amenable but not elementarily amenable group found.

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- Sections of g along sufficiently long words will then have small word length, allowing induction.
- Nekrashevych formalized the notion in a way that makes sense for any self-similar group.

The nucleus

- A self-similar group $G \leq \mathrm{Aut}(T_X)$ is contracting if there is a finite automaton $\mathcal{N} \subseteq G$ such that, for all $g \in G$, there is $k \geq 0$ with $g|_{X^k} \subseteq \mathcal{N}$.

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- In the classical approach the nucleus is a subset of those elements whose sections do not contract by the factor λ.
- Contracting groups include the Grigorchuk group, Gupta-Sidki groups, GGS-groups, Šunić groups associated to polynomials, automaton spinal groups, the basilica group and the Hanoi towers group on 3-pegs.

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- The C^*-algebra is a completion of the complex Nekrashevych algebra.
- The algebraic version generalizes Leavitt algebras.
- They have also been studied by Clark, Exel, Pardo, Sims, Starling, the authors and others.

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- It is a finitely presented simple K-algebra acting faithfully on KX^ω.

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  2. $x^*g = g|_{g^{-1}(x)}(g^{-1}(x))^*$ for $g \in G$, $x \in X$.
- The Nekrashevych algebra $N_K(G,X)$ of G is the K-algebra generated by G and the x, x^* with $x \in X$ subject to the Leavitt algebra relations and the above relations.

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- The groupoid of a self-similar group satisfies the conditions for simplicity of Hausdorff groupoid algebras.
- But these groupoids are rarely Hausdorff, so they are good test cases for understanding simplicity phenomena.

Previous results

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- Nekrashevych (unpublished) showed the Nekrashevych algebra of the Grigorchuk-Erschler group is simple over no field.

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- The algebra depends only on $\mathcal{N}$ and not G.

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- There is a theoretical description of simplicity that underlies the algorithmic result.
- This theoretical description is applicable to many infinite families of contracting groups.
- For any finite set $\mathcal{P}$ of primes, we construct a contracting group G with $N_K(G, X)$ simple over precisely fields K of characteristic not in $\mathcal{P}$.

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- $\mathbb{Z}_p^n$ fixes the first letter of each word and $v|_0 = M_f v$, $v|_{p-1} = \sigma^{v_n}$ and $v|_j = \mathrm{Id}$, else, for $v = (v_1, \ldots, v_n) \in \mathbb{Z}_p^n$.

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- Šunić gave necessary and sufficient conditions for $G_{p,f}$ to be a p-group of intermediate growth.

Šunić groups: continued

Theorem (BS, Szakács)

Let $G_{p,f}$ be a Šunić group with $\deg f = n$.

1. *If $n = 1$, the Nekrashevych algebra of $G_{p,f}$ is simple over all fields.*

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- The Grigorchuk-Erschler group G_{2,x^2+1} has a non-simple algebra because the companion matrix fixes $(1,1)$.

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- We also gave characterizations of simplicity for self-replicating spinal automaton groups and multi-edge spinal groups.

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Theorem (BS, Szakács)

The algebra of a congruence-free inverse monoid has a unique maximal ideal containing Exel's tight ideal, called the singular ideal.

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- If G is contracting, we may assume that a is supported on the nucleus $\mathcal{N}$, a finite set.

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- The obstruction to solving the general case is bounding the max word length in the support of a singular element of KG not belonging to $(1 - \sum_{x \in X} xx^*)$.

The end

Thank you for your attention!