Injective modules over a Leavitt path algebra

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The project

The goal: To study the module category of a Leavitt path algebra.
joint project with G.Abrams and A.Tonolo

G.Abrams, P.Ara, M.Sines Molina *Leavitt Path Algebras* , Lecture Notes in Mathematics, Springer/Verlag 2017

Outline

1. Definitions of a LPA and examples.
2. On the module category of a LPA: main results
3. The injectives modules over a LPA.

The algebra $L_K(E)$

Let $E = (E^0, E^1, s, r)$ be a direct graph. The <u>*extended graph of E*</u> is the graph $\hat{E} = (E^0, E^1 \cup (E^1)^*, s', r')$, with $(E^1)^* = \{e^* \mid e \in E^1\}$, $r'_{|_{E^1}} = r$, $s'_{|_{E^1}} = s$, $r'(e^*) = s(e)$, $s'(e^*) = r(e)$.

Example: Let E be the graph
$$\begin{tikzcd} \bullet^{v} \arrow[loop left, "c"] \arrow[r, "d"] & \bullet^{w} \end{tikzcd}$$

The extended graph $\hat{E}$ is:
$$\begin{tikzcd} \bullet^{v} \arrow[loop left, "c"] \arrow[loop below, dotted, "c^*"] \arrow[r, bend left, "d"] & \bullet^{w} \arrow[l, bend left, dotted, "d^*"] \end{tikzcd}$$

The algebra $L_K(E)$

The Leavitt path algebra $L_K(E)$ of E in the field K is the K-path algebra $K\hat{E}$ modulo the relations:

- $e^* e' = \delta_{e,e'} r(e)$ for any $e, e' \in E^1$
- $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for any $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$.

Example: Let E: $\begin{tikzcd} \bullet^{v} \arrow[loop left, "c"] \arrow[r, "d"] & \bullet^{w} \end{tikzcd}$; $\hat{E}$: $\begin{tikzcd} \bullet^{v} \arrow[loop left, "c"] \arrow[loop below, dotted, "c^*"] \arrow[r, bend left, "d"] & \bullet^{w} \arrow[l, bend left, dotted, "d^*"] \end{tikzcd}$

The Leavitt path algebra $L_K(E)$ is the path algebra $K\hat{E}$ modulo the relations:

$$c^* c = v, \; d^* d = w, \; d^* c = 0, \; c^* d = 0, \; cc^* + dd^* = v$$

Well-known examples

$\mathbf{E}$	$\mathbf{L_K(E)}$
$\begin{tikzcd} \bullet^{v} \arrow[loop left, "e"] \end{tikzcd}$	$K[x, x^{-1}]$
$\bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \dots \xrightarrow{e_{n-2}} \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$	$M_n(K)$
$\begin{tikzcd} \bullet^{v} \arrow[loop left, "e_1"] \arrow[loop above, "e_2"] \arrow[loop right, dotted, "e_i"] \arrow[loop below, "e_n"] \end{tikzcd}$	The Leavitt algebra $L_K(1, n)$ $(L_K(E) \cong L_K(E)^n$ for $n \geq 1)$
$\begin{tikzcd} \bullet^{v} \arrow[loop left, "c"] \arrow[r, "d"] & \bullet^{w} \end{tikzcd}$	$K < X, Y \mid XY = 1 >$

Strategies and techniques

- In general $L_K(E)$ has some specified algebraic property $\Leftrightarrow$ E has some specified graph-theoretic property (for instance: description in terms of the graph of those Leavitt path algebras which are prime, finite dimensional, noetherian, artinian...)
- We aim to describe the module category of $L_K(E)$ in terms of the graph E.
- One can act with sequences of operations on the graphs preserving Morita equivalences between $L_K(E)$ and $L_K(F)$ in order to reduce from E to a simpler graph F. So we can study modules over $L_K(F)$ instead of $L_K(E)$.
- In the language of symbolic dynamics, they correspond to *flows of standard transformations*

Strategies and techniques: LPA as universal localizations

Theorem: (Ara-Brustenga, 2010) The Leavitt path algebra $L_K(E)$ is a *perfect left localization* of a path algebra.

Indeed, denoted by $\overline{E}$ the opposite graph of E, the inclusion $P(\overline{E}) \hookrightarrow L_K(E)$ is a ring epimorphism. It is the universal localization w.r.t a suitable set of maps between fin. gen. projective $P(\overline{E})$-modules. And $L_K(E)$ is flat as $P(\overline{E})$-module.

Corollary: The category of finitely presented $L_K(E)$-modules is equivalent to a quotient category of the finitely presented $P(\overline{E})$-modules (w.r.t a suitable Serre subcategory).

Properties of $L_K(E)$

- $L_K(E)$ is unital if and only if $|E^0|$ is finite. In this case, $1_{L_K(E)} = \sum_{v \in E^0} v$. From now on assume E finite and K algebraically closed.
- The algebra $L_K(E)$ is hereditary.
- Any Leavitt path algebra is a Bézout ring, i.e. every finitely generated one-sided (left and right) ideal of $L_K(E)$ is principal.

Modules over a LPA

Let E be a finite graph and $L_K(E)$ the associated LPA.

- **Projective modules**: By the Bézout property, they are direct sums of cyclic ideals.
- **Simple modules** : a complete classification does not exist. But for any closed path c in the graph and any $0 \neq k \in K$, there are finitely presented simple modules with relevant properties. They are the *Chen simple modules* associated to the closed path c.
- **Injective modules**: unknown. We construct a family of injective modules as the injective envelopes of Chen simple modules. They are *Prüfer-type* modules, or *formal power series* modules.

Chen simple modules: examples

Let $\begin{tikzcd} \bullet^v \arrow[loop left, "c"] \arrow[r, "d"] & \bullet^w \end{tikzcd}$.

- for $k = 1$, V_c^1 is the K-vector space generated by the *infinite path* $c^\infty = cccc \dots$. So $V_c^1 = \{\lambda c^\infty \mid \lambda \in K\}$
- V_c^1 has a structure of left $L_K(E)$-module, it is simple, finitely presented and isomorphic to $L_K(E)/L_K(E)(c-1)$
- In general, V_c^k is simple, finitely presented and isomorphic to $L_K(E)/L_K(E)(c-k)$

Let
$$\begin{tikzcd} \bullet^u \arrow[loop left, "f"] \arrow[r, "g"] & \bullet^v \arrow[loop above, "c"] \arrow[r, "d"] & \bullet^w \end{tikzcd}.$$

- V_f^1 is the K-vector space generated by the *infinite path* f^∞. It is a simple left $L_K(E)$-module, isomorphic to $L_K(E)/L_K(E)(f-1)$.
- V_c^1 is the K-vector space generated by the *infinite paths* c^∞ and $f^i gc^\infty$, for any $i \geq 0$. In this case $\dim_K V_c^1 = \infty$.
- V_c^1 has a structure of left $L_K(E)$-module, it is simple, finitely presented and isomorphic to $L_K(E)/L_K(E)(c-1)$.

Chen simple modules

For any closed path c, the Chen simple module V_c^k comes from infinite paths ending in c^∞, and it is $L_K(E)/L_K(E)(c-k)$.

If w is a sink vertex, then $L_K(E)w$ is a projective and simple module. It can be regarded as the Chen simple associated to the closed path w.

Finitely presented simple modules

Theorem: [Ara-Rangaswamy] All the simple modules over $L_K(E)$ are finitely presented $\Leftrightarrow$ all the simple modules are Chen simple modules associated to closed paths in E $\Leftrightarrow$ any vertex in E is the base of at most one cycle.

Examples:
$$\begin{tikzcd} \bullet^v \arrow[loop left, "c"] \arrow[r, "d"] & \bullet^w \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \bullet^u \arrow[loop left, "f"] \arrow[r, "g"] & \bullet^v \arrow[loop above, "c"] \arrow[r, "d"] & \bullet^w \end{tikzcd}.$$

Prüfer modules over LPA

Fix a closed path c in E and the associated Chen simple module V_c^k. Then $\mathrm{Ext}^1(V_c^k, V_c^k) \neq 0$.

So, for any $n > 0$, we construct a uniserial module M_n of length n with all the composition factors isomorphic to V_c^k.
It is $M_i \leq M_j$ for $i \leq j$.
The direct limit of the M_n's is called the *Prüfer* module U_c^k over $L_K(E)$

Proposition For any closed path $c \in E$ and for any $0 \neq k \in K$, there exists a *Prüfer module* U_c^k, which is indecomposable, uniserial and artinian. All its composition factors are isomorphic to V_c^k.

Problem: Is U_c^k an injective module?

Strategy for the injectives

For suitable c's, U_c^k is pure-injective. So it is injective if the Baer's criterion is tested on the finitely generated left ideals of $L_K(E)$.

Since the finitely generated left ideals of $L_K(E)$ are principal, the Baer's criterion is equivalent to test the division of the elements of the Prüfer module U_c^k by the elements of $L_K(E)$.

So, since any LPA is Bézout, the notion of divisibility plays a role for the injectivity, as in the commutative setting of a PID.

And we have a division algorithm for U_c^k

When a Prüfer module is injective

Theorem: Let E be a finite graph and let c be a closed path in E. Let U_c^k be the Prüfer module associated to c. Then U_c^k is injective if and only if c is a maximal cycle (i.e. no cycles in E connect to c).

In case U_c^k is injective, then :

1. U_c^k is the injective envelope of the Chen simple module V_c^k;
2. $\mathrm{End}_{L_K(E)}(U_c^k)$ is isomorphic to the ring $K[[x]]$ of formal power series in x.

What we need

- General results on uniserial modules
- A division algorithm to solve equations in U_c^k.
- Morita equivalences to reduce E to a simpler graph F, using graph reduction operations.

A particular case: the Jacobson Algebra

Consider the *Jacobson Algebra* $R = K < X, Y | XY = 1 >$. It is isomorphic to $L_K(\mathcal{T})$, where $\mathcal{T}$ is the Toeplitz graph

$$\begin{tikzcd} \bullet^{v} \arrow[loop left, "c"] \arrow[r, "d"] & \bullet^{w} \end{tikzcd}$$

A complete list of non-isomorphic simple left modules is given by

- the projective module $L_K(\mathcal{T})w$ associated to the sink w
- the Chen simple modules V_c^k.

The injective envelopes of the simple modules

Since c is a maximal cycle, then the Prüfer module asociated to any Chen simple modules is injective. So it is the injective envelope of the simple itself.

For the simple $L_K(\mathcal{T})w$, we have that $\mathrm{Ext}^1(L_K(\mathcal{T})w, L_K(\mathcal{T})w) = 0$, so we cannot repeat the Prüfer construction.

New strategy: any element of $L_K(\mathcal{T})w$ can be written uniquely as $k_{-1}w + \sum_{i=0}^{n} k_i c^i d$, with $k_i \in K$. Intuitively, we build the injective envelope of $L_K(\mathcal{T})w$ using "formal series" (as the injective envelope of $K[x]$ is built using formal power series $K[[x]]$).

Definition: Let Y denote the K-space whose elements are "formal series" of the form

$$Y := \{k_{-1}w + k_0 d + k_1 cd + \cdots + k_i c^i d + \cdots \mid k_i \in K\}.$$

- The K-space Y has a natural structure as a left $L_K(\mathcal{T})$-module
- $L_K(\mathcal{T})w$ is the $L_K(\mathcal{T})$-submodule of Y consisting of those elements for which $k_i = 0$ for all $i > N$ for some $N \in \mathbb{N}$, i.e., $L_K(\mathcal{T})w$ consists of the "standard polynomials" in Y.

The injective envelope of $L_K(\mathcal{T})w$

We use again the Bézout property of $L_K(\mathcal{T})$ and a division algorithm to show that:

- $L_K(\mathcal{T})w$ is essential in Y.
- Y is the injective, so in particular it is the injective envelope of $L_K(\mathcal{T})w$

Theorem: The $L_K(\mathcal{T})$-module $C = Y \oplus (\oplus U_c^k)$ is a minimal injective cogenerator for $L_K(\mathcal{T})$.
A left $L_K(\mathcal{T})$-module M is injective if and only if M is isomorphic to a direct summand of a direct product of copies of C.

Problem: Is it possibile to generalize this approach to other LPAs?

LPAs where all the simples are Chen

Theorem: [Ara-Rangaswamy] All the simple modules over $L_K(E)$ are finitely presented $\Leftrightarrow$ all the simple modules are (twisted) Chen simple modules associated to simple closed paths in E $\Leftrightarrow$ any vertex in E is the base of at most one cycle.

Example: $\begin{tikzcd} \bullet^v \arrow[loop left, "e"] \arrow[r, "d"] & \bullet^w \end{tikzcd}$ or $\begin{tikzcd} \bullet^u \arrow[loop left, "f"] \arrow[r, "g"] & \bullet^v \arrow[loop above, "c"] \arrow[r, "d"] & \bullet^w \end{tikzcd}$.

LPAs where all the simples are Chen

Theorem: Let c be a simple closed path. The injective envelope of V_c^k is:

- The the Prüfer module U_c^k, if c is a maximal cycle.
- The "formal power series" module $\hat{U}_c^k$, if c is not maximal.

Some details: Any Prüfer module U_c^k is generated by a family $\{\alpha_i\}_{i \in \mathbb{N}}$, such that $(c-k)\alpha_i = \alpha_{i-1}$.

$\hat{U}_c^k$ is the left $L_K(E)$-module of the "formal power series" with terms $pc^n\alpha_i$, for p real path and $n \geq 0$.

An example

Consider $L_K(E)$ where E is $\begin{tikzcd} \bullet^u \arrow[loop left, "f"] \arrow[r, "g"] & \bullet^v \arrow[loop above, "c"] \arrow[r, "d"] & \bullet^w \end{tikzcd}$.

The indecomposable injectives in $L_K(E)$ are:

- The injective envelopes of the simples V_f^k: the Prüfer modules U_f^c (f is a maximal cycle).
- The injective envelope of the simple $L_K(E)w$: the module of the "formal series" with terms $f^m gc^n dw$, for $m, n \geq 0$
- The injective envelopes of the simple V_c^k: the modules of "formal series" with terms $f^m gc^n \alpha_i$, for $m, n, j \geq 0$