# Partial actions and Galois theory of commutative rings

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Cimpa School - From Dynamics to Algebra and Representation Theory and Back

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# Contents

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in which they laid the foundations for separable extensions of commutative rings and defined the Brauer group of a commutative ring.

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We say that $R^G \subseteq R$ is a Galois extension, if there are $x_i, y_i \in R, 1 \leq i \leq n$ such that

$$\sum_{1 \leq i \leq n} x_i g(y_i) = \delta_{1,g}, \ \ g \in G.$$

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where:

  - T is $G$-strong if for any $g, h \in G$, the restrictions of $g, h$ to T are equal if and only if $g(t)e = h(t)e$, $t, e \in T$ and $e$ idempotent.
  - $H_T = \{h \in G \mid h(t) = t, t \in T\}$

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- A seven terms exact sequence.

The seven terms-exact sequence reads as follows.

$$0 \to H^1(G,\mathcal{U}(R)) \mapsto \operatorname{Pic}(R^G) \mapsto \operatorname{Pic}(R)^G \mapsto H^2(G,\mathcal{U}(R)) \mapsto B(R/R^G) \mapsto$$
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Hence

- **Hilbert's 90thTheorem.** If $\mathrm{Pic}(R^G) = \{0\}$, then $H^1(G,\mathcal{U}(R)) = \{0\}$.
- **Crossed Product Theorem.** If $\mathrm{Pic}(R) = \{0\}$, there is a group isomorphism $H^2(G,\mathcal{U}(R)) \simeq B(R/R^G)$.

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- D. Crocker, I. Raeburn, D. Williams, Equivariant Brauer and Picard groups and a Chase-Harrison-Rosenberg exact sequence, J. Algebra 307(1) (2007) 397–408.

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Let $G$ be a group with identity $e$ and $X$ be a set, a *partial action* of $G$ on $X$ is a pair $\alpha = (\alpha_g, X_g)_{g\in G}$ such that for $g, h \in G$ :

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- When $X_g = X$ for any $g \in G$, we recover the notion of (global) action of $G$ on $X$.

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- $M_{-t} = \{x \in M \mid t \in (a_x, b_x)\}$;
- $\alpha_t : M_{-t} \ni x \mapsto \gamma_x(t) \in M_t$.

Then $\alpha = \{\alpha_t : M_{-t} \to M_t\}_{t \in \mathbb{R}}$ is a partial action.

The notion of partial action of a group appeared in the context of $C^*$-algebras in

- R. Exel, Circle actions on $C^*$-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequences, *J. Funct. Anal.* **122** (1994), (3), 361 - 401.
- K. McClanaham, $K$-theory for partial crossed products by discrete groups, *J. Funct. Anal.*, **130** (1995), 77–117.

and has been adapted in several contexts. For instance:

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- Si $X$ is a $C^*$-algebra then $X_g$ are closed ideals and the maps $\alpha_g$ are $*$-isomorphisms.

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$$R_1 = R, \quad R_g = k \times k \times \{0\}, \quad R_{g^2} = k \times \{0\} \times k, \quad R_{g^3} = \{0\} \times k \times k$$

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is a unital partial action of $G$ on $R$. Moreover

$$R^\alpha = \{(x, x, x) \mid x \in k\} \simeq k$$

and $\{x_i, y_i\}_{1 \leq i \leq 3}$, where $x_i = y_i = e_i$ is a partial Galois corrdinate system.

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Thus, for a generalization of the sequence a cohomology theory based on partial group actions and a parcial action on a semigroup analog to $\mathrm{Pic}(R)$ are needed.

## DK

M. Dokuchaev, M. Khrypchenko, Partial cohomology of groups, J. Algebra 427 (2015) 142-182.

An $n$-cochain is a map $f : G^n \to R$, such that

$$f(g_1, \ldots, g_n) \in \mathcal{U}(R1_{g_1}1_{g_1g_2}\ldots 1_{g_1g_2\ldots g_n}),$$

for all $n \in \mathbb{N}$.

**Denote:** $C^n(G, R)$ the set of $n$-cochains and $C^0(G, R) = \mathcal{U}(R)$.

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**Denote:** $C^n(G,R)$ the set of $n$-cochains and $C^0(G,R) = \mathcal{U}(R)$.

$C^n(G,R)$ is an abelian group under the point-wise multiplication. Its identity is

$$(g_1, g_2, \ldots, g_n) \mapsto 1_{g_1}1_{g_1g_2}\ldots 1_{g_1g_2\ldots g_n}.$$

# The coboundary homomorphism

Let $\delta^n\colon C^n(G,R) \to C^{n+1}(G,R)$ defined by

$$(\delta^n f)(g_1,\dots,g_{n+1}) = \alpha_{g_1}\left(f(g_2,\dots,g_{n+1})1_{g_1^{-1}}\right)\prod_{i=1}^{n} f(g_1,\dots,g_ig_{i+1},\dots,g_{n+1})^{(-1)^i}$$
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$\delta^n$ is a homomorphism such that

$$\delta^{n+1}\circ\delta^n(f) = \mathrm{id}_{C^{n+2}(G,R)},\quad f\in C^n(G,R)$$

Set $Z^n(G,R) = \ker \delta^n$ and $B^n(G,R) = \operatorname{im} \delta^{n-1}$ and

$$H^n(G,R) = \frac{Z^n(G,R)}{B^n(G,R)}, \qquad H^0(G,R) = Z^0(G,R)$$

the group of partial $n$-cohomologies.

# The two cocycles $Z^2$



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where $[A] = [B]$ if there is $P$ f.g.p such that

$$A \otimes B^{\mathrm{op}} \simeq \mathrm{End}_R(P).$$

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$$\mathcal{B}(S/R) = \{[A] \in \mathcal{B}(R) \mid [A \otimes_R S] = [S]\}$$

# Crossed products



For $\alpha = (\alpha_g, R_g)_{g\in G}$ partial action and $\omega \in Z^2$ one sets

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Then $R *_{\alpha,\omega} G$ is a $G$-graded ring and

**Paques and Sant'Ana (2010)**

$[R *_{\alpha,\omega} G] \in B(R/R^{\alpha})$ provided that $R^{\alpha} \subseteq R$ is a partial Galois extension.

# The Picard semigroup



Recall

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In order to work with partial actions is convenient to use the following.

**Definition**

We set $\mathrm{PicS}(R) = \{[P] \mid P \text{ is f.g.p and } P_{\mathfrak{p}} \simeq R_{\mathfrak{p}} \ \vee \ P_{\mathfrak{p}} = \{0\} \ \forall \, \mathfrak{p} \in \mathrm{Spec}(R)\}$.

# The Picard semigroup



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$$\mathrm{PicS}(R) = \bigcup_{e \in R, e = e^2} \mathrm{Pic}(Re).$$

# A partial action on $\mathrm{PicS}(R)$



Let $\alpha : (\alpha_g, D_g)_{g\in G}$ be a partial action of $G$ on $R$.

Set $X_g = [D_g]\mathrm{PicS}(R) = \mathrm{PicS}(D_g)$ and $\alpha_g^* : X_{g^{-1}} \ni [E] \mapsto [E_g] \in X_g$,

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$E_g = E$ as sets with $R$-module structure

$$r \bullet x = \alpha_g(r1_{g^{-1}})x.$$

# The seven-terms exact sequence


- M. Dokuchaev, A. Paques and H. Pinedo, Partial Galois cohomology and related homomorphism, *Quart. J. Math.* 70 (2019), 737-766.
- M. Dokuchaev, A. Paques, H. Pinedo, and I. Rocha, Partial Generalized crossed products and a seven-term exact sequence, *J. Algebra* 572 (2021), 195-230.

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Let $R^{\alpha} \subseteq R$ be a partial Galois extension, then there is an exact sequence

$$0 \to H^1(G,\mathcal{U}(R)) \mapsto \mathrm{Pic}(R^{\alpha}) \mapsto \mathrm{PicS}(R)^{\alpha^*} \mapsto H^2(G,\mathcal{U}(R)) \mapsto B(R/R^G) \mapsto H^1(G,\mathrm{PicS}(R)) \mapsto H^3(G,\mathcal{U}(R)).$$

Then

- **Hilbert's 90th Theorem for partial actions.** If $\mathrm{Pic}(R^{\alpha}) = \{0\}$, then

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- **Crossed Product Theorem for partial ations.** If $\mathrm{Pic}(R) = \{0\}$ the map
$$H^2(G, \mathcal{U}(R)) \ni \mathrm{cls}(f) \mapsto [R_{\alpha,f} G] \in B(R/R^G).$$
is a group isomorphism.

# Contents

# The Harrison group



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**Harrison group**: The set $\mathcal{H}(G, R) = \{[S] \mid R \subseteq S \text{ is Galois}\}$ is an abelian group with operation

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$$[S][S'] = [(S \otimes_R S')^{\delta_G}],$$

being $\delta_G = \{(g, g^{-1}) \mid g \in G\}$ and $G = G \times G/\delta_G$ acts on via

$$g\left(\sum x_i \otimes y_i\right) = \sum g(x_i) \otimes y_i$$

The identity element is $E_G(R) = \bigoplus_{g\in G} Re_g$ with $G$-action

$$h\left(\sum_{g\in G} r_g e_g\right) = \sum_{g\in G} r_g e_{hg}.$$

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While the inverse of $[S]$ is $[S^*]$ where $S^* = S$ ans sets and the $G$-action id

$$g \cdot s = g^{-1}(s).$$

The idea is to explore such a construction due to D. K. Harrison in the context of partial Galois extensions.

## BCMPP

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**Theorem**: Let $S$ be a ring, $G$ a finite group $H \leq G$, $\alpha_G$ a unital partial action of $G$ on $S$ such that:

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- $H$ is normal in $G$.

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- $H$ is normal in $G$.

Then $\alpha_G$ induces a unital partial action $\alpha_{G/H}$ of $G/H$ on

$$S^{\alpha_H} = \{s \in S \mid \alpha_h(s1_{h^{-1}}) = s1_h,\ h \in H\}.$$

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**Theorem.** Let $G$ be a finite abelian group and $R$ a commutative algebra. Then $\left(H_{par}(G,R), *_{par}\right)$ is a commutative inverse semigroup.

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**Theorem.** Let $G$ be a finite abelian group and $R$ a commutative algebra. Then $\left(H_{par}(G, R), *_{par}\right)$ is a commutative inverse semigroup. Thus

$$H_{par}(G, R) = \bigcup_{e \in E} H_e(G, R)$$

is a union of abelian groups, indexed by the idempotents of $H_{par}(G, R)$.

# $\mathcal{H}(G,R)$ and $H_{E_G(R)}(G,R)$



It is clear that

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Moreover,

**Proposition**

Let $\alpha = \{\alpha_g : S1_{g^{-1}} \to S1_g\}_{g\in G}$ a partial action such that $S \supseteq R$ is a partial Galois extension. If $\mathrm{ann}_R(1_g) = \{0\}$, for any $g \in G$ then $\lfloor S, \alpha_G \rfloor \in H_{E_G(R)}(G,R)$.

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## Example:

Let $S = k \times k \times k$ and $G = \langle g \mid g^4 = 1 \rangle$. The family $\alpha = \{\alpha_x : R_{x^{-1}} \to R_x\}_{x \in G}$ where:

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is partial Galois extension of $R = \{(x, x, x) \mid x \in k\}$ such that

$$\lfloor S, \alpha_G \rfloor \in H_{E_G(R)}(G, R) \setminus \mathcal{H}(G, R).$$

# Reduction to cyclic groups



Assume $G = G_1 \times G_2 \times \cdots \times G_n$ product of cyclic groups, then there is semigroup homomorphism

$$\prod_{i=i}^{n} H_{par}(G_i, R) \simeq H_{par}(G, R).$$

# Contents

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- Hopf Galois Theory.
- Galois Lie rings theory, Galois birings theory.
- Galois descent theory.

# Thanks