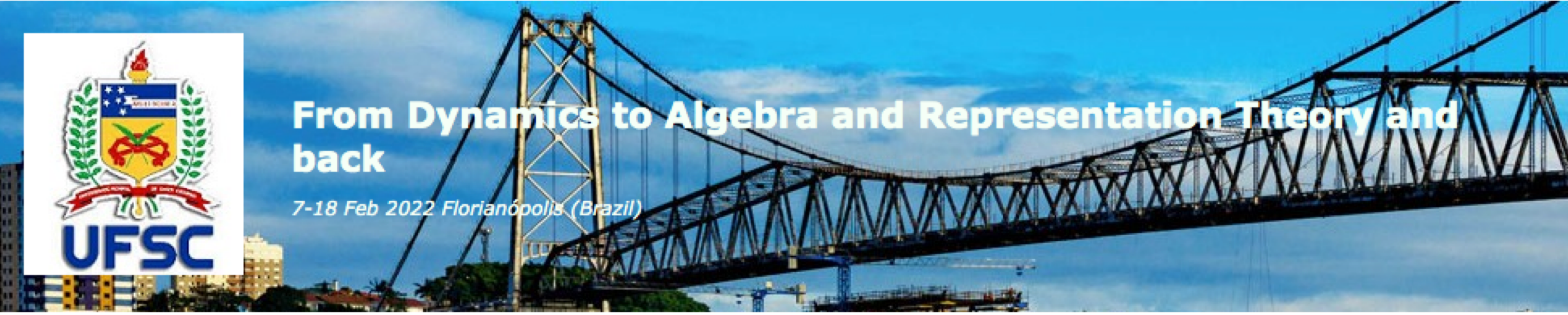

Natural families in evolution algebras

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What we mean by evolution algebra

K-Algebra → A

Basis → $B = \{e_i \mid i \in \Lambda\}$ $\quad$ $e_i e_j = 0$ whenever $i \neq j$.

Natural basis

Product → $e_i^2 := e_i e_i = \sum_{k \in \Lambda} \omega_{ki} e_k$

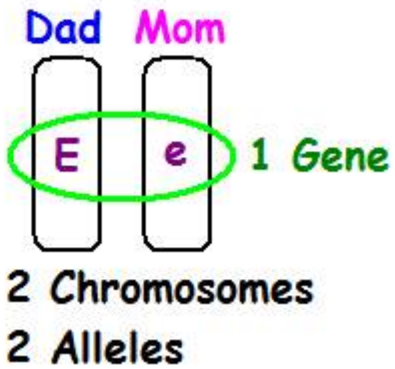
Structure constants

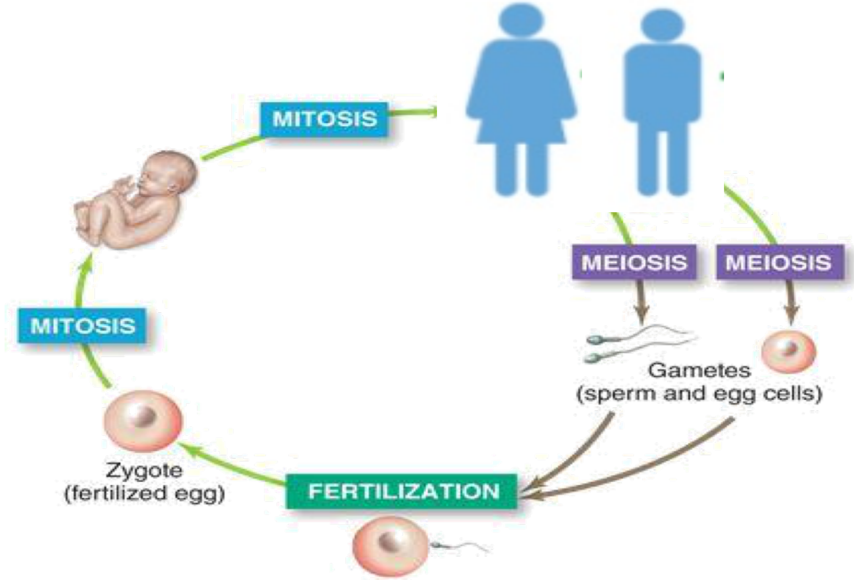
- The matrix $M_B := (\omega_{ki})$ is said to be the **structure matrix of** A **relative to** B.

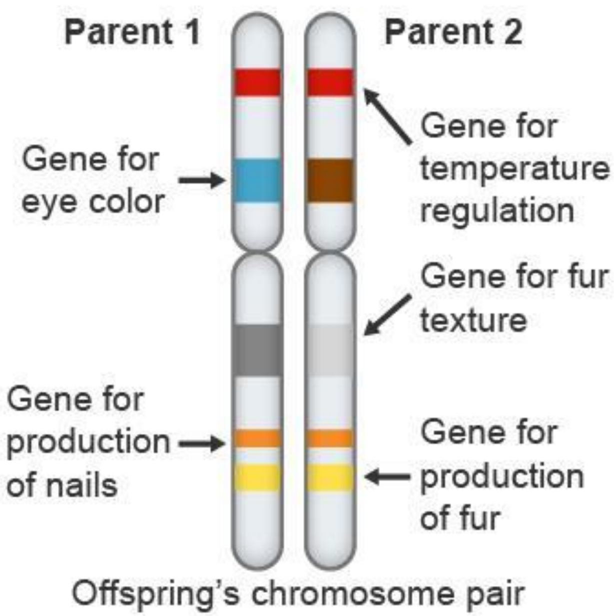
> Jianjun P. Tian
> *Evolution algebras and their applications.*
> Lecture Notes in Mathematics **1921**, Springer-Verlag, Berlin, 2008.

In this book a new and outstanding type of genetic algebra was introduced to model the non-Mendelian genetics: the so-called evolution algebras.

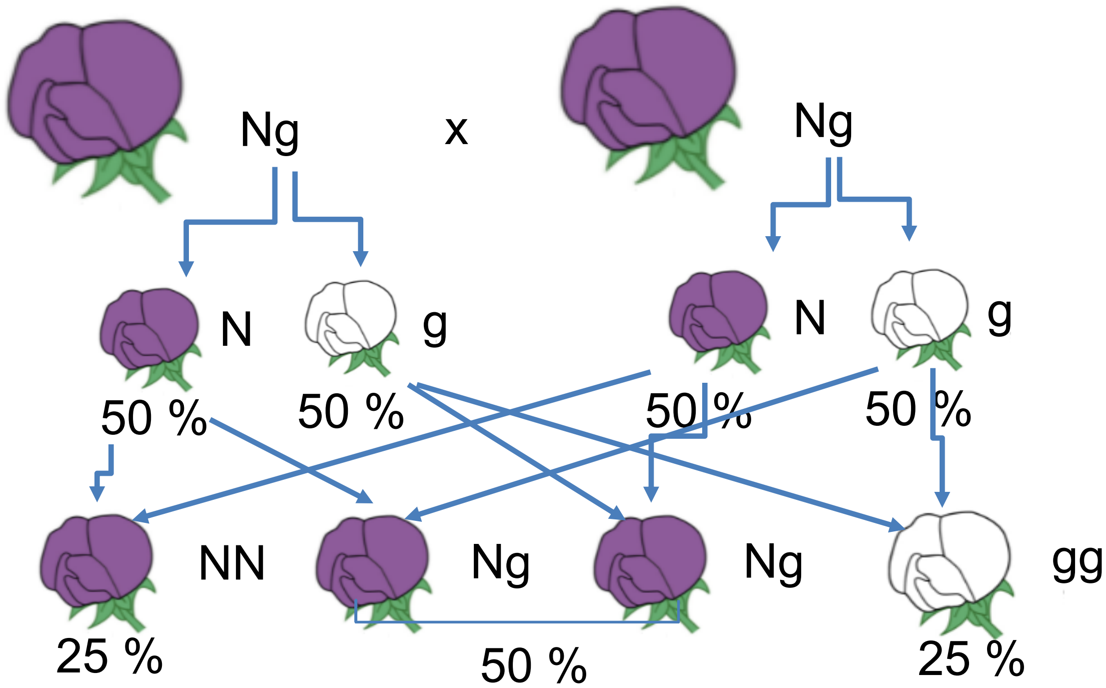
Mathematical formulation of the Mendel's second law







Mathematical formulation of the Mendel's second law



Multiplication table

- **Example: Zygotic algebra (Reed 1997)**

Two alleles → N, g.

Three genotypes → NN, Ng, gg.

Algebra → B={NN, Ng, gg}

	NN	Ng	gg
NN	NN	$\frac{1}{2}(NN+Ng)$	Ng
Ng	$\frac{1}{2}(NN+Ng)$	$\frac{1}{4}(NN+gg)+\frac{1}{2}Ng$	$\frac{1}{2}(gg+Ng)$
gg	Ng	$\frac{1}{2}(gg+Ng)$	gg

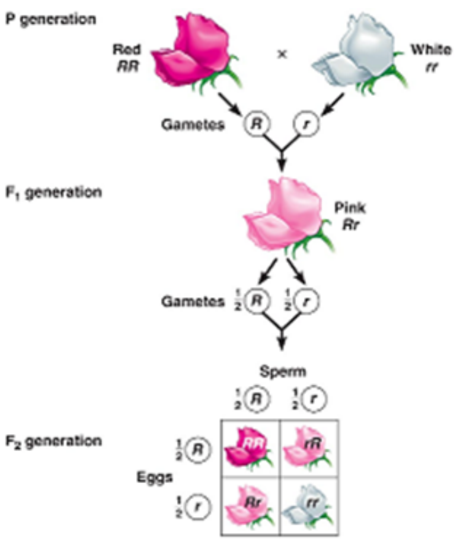
This is not an evolution algebra

What we mean by evolution algebra

In the asexual inheritance,

- $a_i a_j$ does not make sense biologically $(a_i a_j = 0)$ $i \neq j$.
- $a_i a_i = a_i^2 = \sum_{k=1}^{n} \gamma_{ki} a_k$. Interpreted as self-replication.

It is called evolution algebra.



CODOMINANCE

Simultaneously orthogonalizable

Corollary 1: Fix a basis B of a vector space V of finite dimension over a field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$ and assume that $\mathcal{F} = \{\langle\cdot,\cdot\rangle_i\}_{i \in I \cup \{0\}}$ is a family of inner products on V whose matrices in B are $M_{i,B}$. Further assume that $M_{0,B}$ is nonsingular. Then $\mathcal{F}$ is simultaneously orthogonalizable if and only if the collection of matrices $\{M_{i,B} M_{0,B}^{-1}\}_{i \in I}$ is commutative and each one of them is diagonalizable.

Yolanda Cabrera Casado, Cristóbal Gil Canto, Dolores Martín Barquero and Cándido Martín González, Simultaneous orthogonalization of inner products over arbitrary fields. https://arxiv.org/pdf/2012.06533.pdf.

How do you know if an algebra is an evolution algebra?

Structure Inner Products

Definition: If A is a commutative algebra over a field $\mathbb{K}$ the product in A can be written in the form

$$xy = \sum_{i \in I} \langle x, y \rangle_i e_i$$

where $\{e_i\}_{i \in I}$ is any fixed basis of A and the inner products $\langle \cdot, \cdot \rangle_i \colon A \times A \to \mathbb{K}$ provide the coordinates of xy relative to the basis. So A is an evolution algebra if and only if the set of inner products $\langle \cdot, \cdot \rangle_i$ is simultaneously orthogonalizable.

Some General Properties

Evolution algebras are commutative and hence flexible.

The direct sum of evolution algebras is an evolution algebra.

Evolution algebras are not power associative

Jordan, alternative or associative algebras.

The quotient algebra A/I with I ideal of A is an evolution algebra.

Moussa Ouattara and Souleymane Savadogo,
Power-associative evolution algebras
https://arxiv.org/abs/1812.09986.

Change of basis

Theorem: Let A be an evolution algebra and let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of A with structure matrix $M_B = (\omega_{ij})$.

- If $B' = \{f_i \mid i \in \Lambda\}$ is a natural basis of A with $P_{B'B} = (p_{ij})$ the change of basis matrices, then
$$M_B(P * P) = 0$$
for every $i \neq j$ with $i, j \in \Lambda$. Moreover
$$M_{B'} = P_{B'B}^{-1} M_B P_{B'B}^{(2)},$$
where $P_{B'B}^{(2)} = (p_{ij}^2)$.

- Assume that $P = (p_{ij}) \in \mathrm{CFM}_\Lambda(\mathbb{K})$ is invertible and satisfies the first above relation. Define $B' = \{f_i \mid i \in \Lambda\}$, where $f_i = \sum_{j \in \Lambda} p_{ji} e_j$ for every $i \in \Lambda$. Then B' is a natural basis and the second above relation is satisfied.

Change of basis

Definition: Let A be an evolution algebra. We say that A has a unique natural basis if the only change of basis matrices are $S_n \rtimes (\mathbb{K}^\times)^n$.

A unique basis

Definition: Let A be an evolution algebra of dimension n. We say that A has *Property* (2LI) if for any different vectors e_i, e_j of a natural basis, the set $\{e_i^2, e_j^2\}$ is linearly independent.

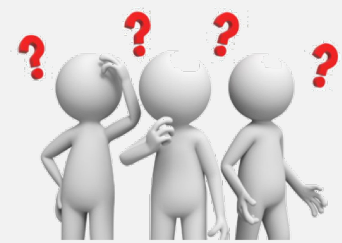
Corollary: Let A be a non-degenerate evolution algebra over $\mathbb{K}$. Then the following assertions are equivalent:

1. A has a unique natural basis.
2. There exists a natural basis B such that for any 2 different vectors u and v of B, u^2 and v^2 are linearly independent.

Evolution subalgebras. Evolution ideals

Fact: There are ideals (and hence subalgebras) of an evolution algebra which are not evolution algebras.

Example: Let A be the evolution algebra with natural basis $B = \{e_1, e_2, e_3\}$ and product $e_1^2 = e_2 + e_3$; $e_2^2 = e_1 + e_2$ and $e_3^2 = -(e_1 + e_2)$. Let



$$I = \{\alpha e_1 + (\alpha + \beta)e_2 + \beta e_3 : \alpha, \beta \in \mathbb{K}\}$$

Then I is an ideal without a natural basis. Therefore:

I is an ideal (and hence a subalgebra) but I is not an evolution algebra.

Definition: An evolution subalgebra (respectively ideal) of an algebra A is a subalgebra (respectively Ideal) provided with a natural basis.

Evolution subalgebras. Evolution ideals

Fact: Evolution subalgebra does not need to be an ideal.

Example: Let A be an evolution algebra with natural basis $B = \{e_1, e_2, e_3\}$ such that $e_1^2 = e_2$, $e_2^2 = e_1$ and $e_3^2 = e_3$. Then, the subalgebra A' generated by $e_1 + e_2$ and e_3 is an evolution subalgebra but it is not an ideal as $e_1(e_1 + e_2) \notin A'$.

Something less restrictive and more algebraically natural

Extension property

- **Fact:** Not every basis of an evolution subalgebra can be extended to a natural basis of the whole algebra.

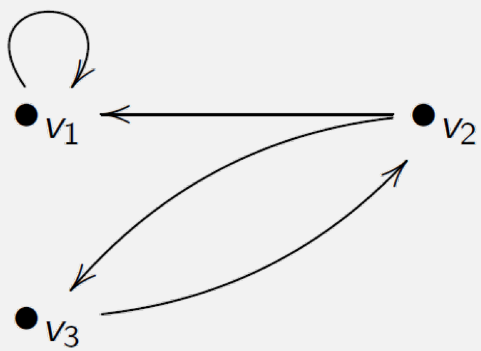
> Example: Let A be an evolution algebra with natural basis $B = \{e_1, e_2, e_3\}$ and multiplication given by $e_1^2 = e_3$, $e_2^2 = e_1 + e_2$ and $e_3^2 = e_3$.
> Let I be the evolution ideal generated by $e_1 + e_2$ and e_3.

- These definitions are less restrictive.

Graph associated

Example: $B = \{e_1, e_2, e_3\}$ natural basis of A with product:

$$e_1^2 = -5e_1, \;\; e_2^2 = 2e_1 - 3e_3, \;\; e_3^2 = -2e_2.$$



Graph G:

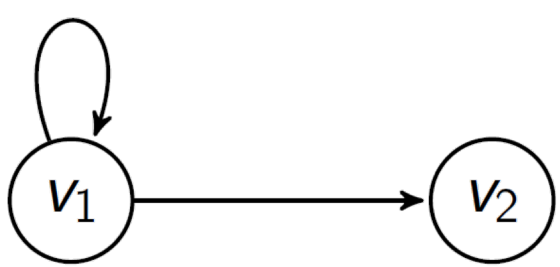
$$\begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Adjacency matrix:

- The graph associated to an evolution algebra depends on the choosen basis.
    - Isomorphic evolution algebras $\not\Rightarrow$ isomorphic graphs.

Example: Let A be the evolution algebra with natural basis $B = \{e_1, e_2\}$ and product given by $e_1^2 = e_1 + e_2$ and $e_2^2 = 0$. Consider the natural basis $B' = \{e_1 + e_2, e_2\}$. Then the graphs associated to the bases B and B' are, respectively

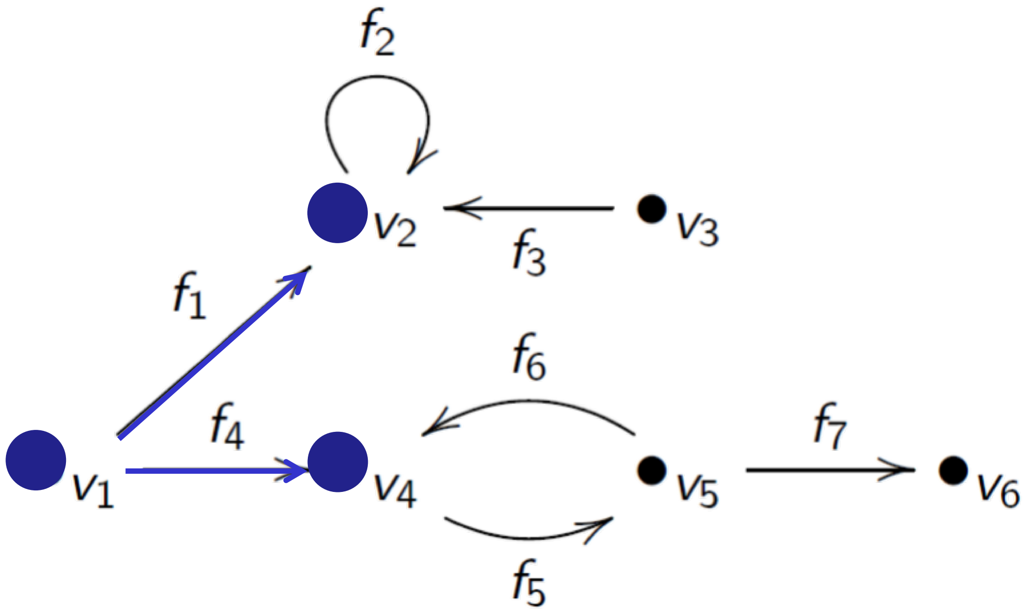
E:



F:

Let A evolution algebra with structure matrix

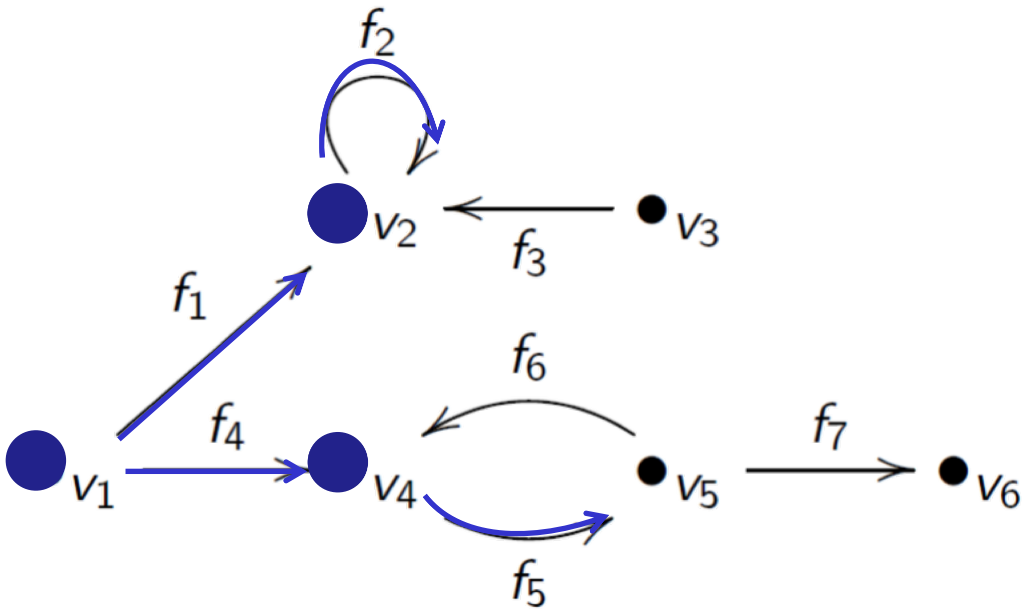
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & 0 \end{pmatrix}$$



$D^1(1) = \{k \in \Lambda \mid e_1^2 = \sum_k \omega_{k1} e_k \text{ with } \omega_{k1} \neq 0\} = \{2, 4\}.$

Descendents

Let A evolution algebra with structure matrix $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & 0 \end{pmatrix}$



$$D^1(1) = \{k \in \Lambda \mid e_1^2 = \textstyle\sum_k \omega_{k1} e_k \text{ with } \omega_{k1} \neq 0\} = \{2, 4\}.$$

$$D^2(1) = \bigcup_{k \in D^1(1)} D^1(k) = \{2, 5\}.$$

Descendents

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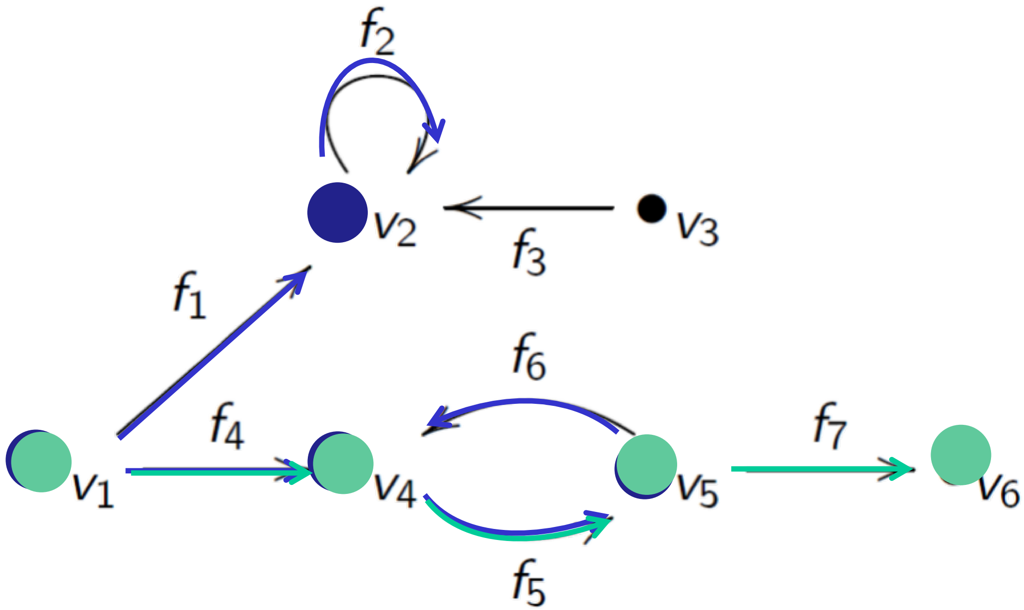


$$D^1(1) = \{k \in \Lambda \mid e_1^2 = \textstyle\sum_k \omega_{k1} e_k \text{ with } \omega_{k1} \neq 0\} = \{2, 4\}.$$

$$D^2(1) = \bigcup_{k \in D^1(1)} D^1(k) = \{2, 5\}.$$

$$D^3(1) = \{2, 4, 6\}. \qquad D(1) = \bigcup_{m \in \mathbb{N}} D^m(1) = \{2, 4, 5, 6\}.$$

$$D^4(1) = \{2, 5\}.$$

Definition: An evolution algebra A is **non-degenerate** if it has a natural basis $B = \{e_i \mid i \in \Lambda\}$ such that $e_i^2 \neq 0$ for every $i \in \Lambda$.

Lemma

A Non-degenerate $\Leftrightarrow$ $\mathrm{ann}(A)=0$

$\mathrm{ann}(A):=\{x \in A \mid xA = 0\}$

$\mathrm{ann}(A)=\mathrm{lin}\{e_i \in B \mid e_i^2 = 0\}$

Does not depend on the basis

It is an evolution ideal of A

Annihilator. Properties

Remark: Let A be an evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis.

- $A/\mathrm{ann}(A)$ is not necessarily a non-degenerate evolution algebra.
- $A_1 := \mathrm{lin}\{e_i \in B \mid e_i^2 \neq 0\}$ is not necessarily a subalgebra of A.

Absorption property. Properties

$$\mathrm{ann}(A/\mathrm{ann}(A)) = \bar{0} \text{ ?}$$

$\boxed{\textbf{No}}$

Absorption property

$$xA \subseteq I \text{ implies } x \in I$$

$$\Updownarrow$$

$$\mathrm{ann}(A/I) = \bar{0}$$

Proposition

Absorption property. Properties

$\operatorname{ann}(A/\operatorname{ann}(A)) = \overline{0}$ **?** **No**

Absorption radical

Intersection of all absorption ideals

Proposition

$$\operatorname{rad}(A) = 0$$

$$\Updownarrow$$

$$\operatorname{ann}(A) = 0$$

$$\Updownarrow$$

A *Non-degenerate*

$$\mathbf{rad}(A/\mathbf{rad}(A)) = \overline{0}$$

Simple evolution algebra

Definition: An algebra A is simple if $A^2 \neq 0$ and 0 is the only proper ideal.

Proposition: Every simple evolution algebra, then A is non-degenerate.

Remark: An evolution algebra A whose associated graph has sinks cannot be simple.

Simple evolution algebra: Characterization

Theorem: Let A be a non-zero evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis. The following conditions are equivalent:

- A is simple.
- If $\mathrm{lin}\{e_i^2 \mid i \in \Lambda'\}$ is an ideal for a non-empty subset $\Lambda' \subseteq \Lambda$, then $A = \mathrm{lin}\{e_i^2 \mid i \in \Lambda'\}$.
- $A = \langle e_i^2 \rangle = \mathrm{lin}\{e_j^2 \mid j \in D(i)\}$ for every $i \in \Lambda$.
- $A = \mathrm{lin}\{e_i^2 \mid i \in \Lambda\}$ and $\Lambda = D(i)$ for every $i \in \Lambda$.

Simple finite evolution algebra

In terms of structure matrix:

Corollary: Let A be a n-dimensional evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis of A. Then A is simple if and only if $|M_B(A)| \neq 0$ and B cannot be reordered in such a way that the corresponding structure matrix is:

$$\begin{pmatrix} W_{m\times m} & U_{m\times(n-m)} \\ 0_{(n-m)\times m} & Y_{(n-m)\times(n-m)} \end{pmatrix}$$

Reducible evolution algebras

Definition: An evolution algebra is called **reducible** if $A = \oplus_{\gamma\in\Gamma} A_\gamma$ where A_γ is a non-zero evolution subalgebra. If A_γ is irreducible, then $A = \oplus_{\gamma\in\Gamma} A_\gamma$ is an **optimal decomposition of A**.

Proposition: Let A be an evolution algebra. The following assertions are equivalent:

- There exists a family of evolution subalgebras $\{A_\gamma\}_{\gamma\in\Gamma}$ such that $A = \oplus_{\gamma\in\Gamma} A_\gamma$.
- There exists a family of evolution ideals $\{I_\gamma\}_{\gamma\in\Gamma}$ such that $A = \oplus_{\gamma\in\Gamma} I_\gamma$.
- There exists a family of ideals $\{I_\gamma\}_{\gamma\in\Gamma}$ such that $A = \oplus_{\gamma\in\Gamma} I_\gamma$.

Reducible evolution algebras

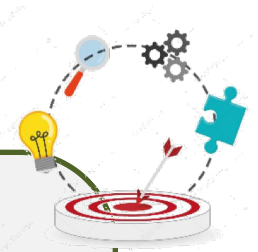
Infinite-dimensional case:

Theorem: Let A be a non-degenerate evolution algebra with a natural basis $B = \{e_i \mid i \in \Lambda\}$. Assume that $A = \oplus_{\gamma\in\Gamma} I_\gamma$, where each I_γ is an ideal of A. Then, there exists a disjoint decomposition of $\Lambda = \sqcup_{\gamma\in\Gamma}\Lambda_\gamma$ such that

$$I_\gamma = \mathrm{lin}\{e_i \mid i \in \Lambda_\gamma\}.$$

Reducible evolution algebras

Finite-dimensional case:



Corollary: A non-degenerate finite dimensional evolution algebra A with natural basis $B = \{e_i \mid i = 1, \dots, n\}$ is reducible if and only B can be ordered in such a way the corresponding structure matrix is

$$\begin{pmatrix} W_{m\times m} & 0_{(n-m)\times(n-m)} \\ 0_{(n-m)\times m} & Y_{(n-m)\times(n-m)} \end{pmatrix}$$

In terms of graph:

Corollary: Let A be a non-degenerate evolution algebra. Then A es irreducible if and only if the associated graph is connected.

Reducible evolution algebras

Theorem: The optimal direct sum decomposition of an evolution algebra A does exist and it is unique whenever the algebra is non-degenerate.

- **Finite-dimensional case:** If A is a finite dimensional evolution algebras (degenerated or not), we get an optimal decomposition through the fragmentation process (decomposition of E into its connected components).

Orthogonality

Definitions:

- Two elements u and v are orthogonal if $uv = 0$.
- A family of vectors C is an orthogonal family if any pair of elements are orthogonal.
- A family of pairwise orthogonal and linearly independent vectors of an evolution algebra which have the extension property will be called an extending natural family.
- Any (linear) subspace E of an evolution algebra generated by an extending natural family is an extending evolution subspace. Such a family will be called an extending natural basis of E.
- The evolution rank of E is defined by $\mathrm{erk(E)} = \dim(\mathrm{E}^2)$.

Natural vectors

Definition: Let A be an evolution algebra, $B = \{e_i\}_{i\in\Lambda}$ a natural basis and $u = \sum_{i\in\Lambda} \alpha_i e_i$ an element of A. The support of u *relative to* B, denoted $\mathrm{supp}_B(u)$, is defined as the set $\mathrm{supp}_B(u) = \{i \in \Lambda \mid \alpha_i \neq 0\}$. If $X \subseteq A$, we put $\mathrm{supp}_B(X) = \cup_{x\in X} \mathrm{supp}_B(x)$.

Theorem: Let A be an evolution $\mathbb{K}$-algebra with natural basis $B = \{e_i\}_{i\in\Lambda}$ and let $u \in A$. Set $\mathrm{supp}(u) = \{i_1, \dots, i_r\}$. Then

1. If $u^2 \neq 0$, then u is a natural vector if and only if $\mathrm{rk}(\{e_{i_1}^2, \dots, e_{i_r}^2\}) = 1$.
2. If $u^2 = 0$, then u is a natural vector if and only if $e_{i_1}^2 = \cdots = e_{i_r}^2 = 0$.

Natural families. Another decomposition

Theorem: Let A be an evolution algebra and let $r = \dim A^2$. Then:

$$A = \mathrm{ann(A)} \oplus \mathrm{E_1} \oplus \ldots \oplus \mathrm{E_r},$$

where $E_1, \ldots, E_r$ are extending evolution subspaces of A satisfying $\mathrm{erk(E_i)} = 1$ for all i and if $i \neq j$, $E_i E_j = 0$, $\dim\,(\mathrm{E_i^2} + \mathrm{E_j^2}) = 2$. Moreover, if A is non-degenerate, the decomposition is unique.

Decomposition

Corollary:
Let A be an evolution algebra and let $B = B_0 \cup B_1 \cup \cdots \cup B_r$ and $B' = B'_0 \cup B'_1 \cup \cdots \cup B'_r$ be two natural bases of A given by two decompositions as in previous theorem, where B_0 and B'_0 are bases of $\mathrm{ann(A)}$. Then, we can reorder the elements of B and B' so that the change of basis matrix has the following block form

$$\begin{pmatrix} * & * & * & \dots & * \\ 0 & * & 0 & \dots & 0 \\ 0 & 0 & * & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & * \end{pmatrix}.$$

Extension property

Proposition: Let A be an evolution algebra and let E be an extending evolution subspace of A with evolution rank one and such that $E \cap \mathrm{ann(A)} = \{0\}$. Let C be a linearly independent orthogonal family of E. Then C can be extended to a natural basis of E, which can be extended to a natural basis of A, if and only if $u^2 \neq 0$ for all $u \in C$.

Gian Carlo Rota, Discrete Thoughts (1953).
The lack of real contact between mathematics and biology is either a tragedy, a scandal or a challenge, it is hard to decide which.