Group-type partial actions of groupoids and a Galois correspondence

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This is a joint work with D. Bagio and A. Sant'Ana

Groupoids

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Given $x, y \in \mathrm{G}_0$ we set

$$\mathrm{G}(x, y) := \{g \in \mathrm{G} : s(g) = x \text{ and } t(g) = y\}.$$

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$\mathrm{G}^{(2)} = \{(g, h) \in \mathrm{G} \times \mathrm{G} \ : \ s(g) = t(h)\}$ denote the set of pairs of G which are composable. It is clear that

$$s(g) = g^{-1}g, \quad t(g) = gg^{-1}, \quad s(gh) = s(h), \quad t(gh) = t(g),$$

for all $x \in \mathrm{G}_0$, $g \in \mathrm{G}$ and $(g, h) \in \mathrm{G}^{(2)}$

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$$x \sim y \text{ if and only if } \mathsf{G}(x, y) \neq \emptyset, \qquad x, y \in \mathsf{G}_0,$$

induces the decomposition $\mathsf{G} = \dot{\bigcup}_{Y \in \mathsf{G}_0/\sim} \mathsf{G}_Y$ of G in connected components. For each equivalence class $Y \in \mathsf{G}_0/\sim$, the set of the objects of G_Y is Y and $\mathsf{G}_Y(x, y) = \mathsf{G}(x, y)$ for all $x, y \in Y$.

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A *transversal* in G for x is a map $\tau : \mathrm{G}_0 \to \mathrm{G}$, $\tau(y) = \tau_y$, such that $\tau_y \in \mathrm{G}(x, y)$ for all $y \in \mathrm{G}_0$ and $\tau_x = x$. We then have bijections

$$\tau : \mathrm{G}(y, z) \to \mathrm{G}(x), \qquad \tau(g) = \tau_z^{-1} g \tau_y, \qquad y, z \in \mathrm{G}_0 .$$

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Now consider the groupoid $H = \mathrm{G}_0^2 \times \mathrm{G}(x)$. We have $H_0 = \mathrm{G}_0$. The composition is given by $(z, u, k)(y, z, l) = (y, u, kl)$.

Proposition 1.1

We have an isomorphism of groupoids ψ between G and H. It is the identity at the level of objects. At the level of morphisms, we have

$$\psi(y,z) : \mathsf{G}(y,z) \to H(y,z), \quad \psi(y,z)(g) = (y,z,\tau(g)).$$

Partial Actions

Definition 2.1

A unital partial action of a groupoid G *on a ring* S *is a family of pairs* $\alpha = (S_g, \alpha_g)_{g \in \mathsf{G}}$, *where* S_g *is a two-sided ideal generated by a central idempotent* 1_g *of* S *and* $\alpha_g : S_{g^{-1}} \to S_g$ *is an isomorphism of rings, that satisfies:*

(i) $1_g 1_{t(g)} = 1_g$ *(or equivalently* $S_g \subset S_{t(g)}$*), for all* $g \in \mathsf{G}$,

(ii) *for each* $x \in \mathsf{G}_0$, $\alpha_x = \mathrm{id}_{S_x}$ *is the identity map of* S_x,

(iii) $\alpha_h^{-1}(S_{g^{-1}} \cap S_h) \subseteq S_{(gh)^{-1}}$, *for all* $(g, h) \in \mathsf{G}^{(2)}$,

(iv) $\alpha_g(\alpha_h(a)) = \alpha_{gh}(a)$, *for all* $a \in \alpha_h^{-1}(S_{g^{-1}} \cap S_h)$ *and* $(g, h) \in \mathsf{G}^{(2)}$.

The items (iii) and (iv) above imply that α_{gh} is an extension of $\alpha_g\alpha_h$. When $\alpha_g\alpha_h = \alpha_{gh}$, for all $(g, h) \in \mathsf{G}^{(2)}$, we say that α is *global*.

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From now on, partial action means unital partial action.

Remark 2.2

Let $\alpha = (S_g, \alpha_g)_{g\in \mathsf{G}}$ be a partial action of a groupoid G on a ring S, $x \in \mathsf{G}_0$ and H a subgroupoid of G. Then:

(i) *the isotropy group $\mathsf{G}(x)$ acts partially on S_x via $\alpha_{\mathsf{G}(x)} := (S_g, \alpha_g)_{g\in \mathsf{G}(x)}$,*

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(ii) *if* $x \in H_0$ *then* $\alpha_{H(x)} := (S_h, \alpha_h)_{h \in H(x)}$ *is a partial action of* $H(x)$ *on* S_x*; in this case* $\alpha_{H(x)}$ *is the restriction of* $\alpha_{\mathsf{G}(x)}$ *to* $H(x)$,

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(iii) *if* $S = \bigoplus_{y \in \mathsf{G}_0} S_y$ *and* H *is a subgroupoid of* G *then we can consider the restriction* $\alpha_H := (S_h, \alpha_h)_{h \in H}$ *of* α *to* H *which is a partial action of* H *on* $S_H := \bigoplus_{z \in H_0} S_z$.

Partial Actions of Group-type

The definition of group-type partial action is due to Bagio, Paques and Pinedo:

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Definition 2.3

A partial action $\alpha = (S_g, \alpha_g)_{g\in \mathsf{G}}$ *of a connected groupoid* G *on* S *is called group-type if there exist and element* $x \in \mathsf{G}_0$ *and a transversal* $\tau = \{\tau_y\}_{y\in \mathsf{G}_0}$ *in* G *for* x *such that*

$$S_{\tau_y^{-1}} = S_x \ \text{ and } \ S_{\tau_y} = S_y, \quad \text{for all } \ y \in \mathsf{G}_0 . \tag{1}$$

Any global groupoid action $\alpha = (S_g, \alpha_g)_{g \in \mathsf{G}}$ of G on S is group-type. In fact, Lemma 1.1 of

• [DP] D. Bagio, A. Paques, *Partial groupoid actions: globalization, Morita theory and Galois theory*, Comm. Algebra 40 (10) (2012), 3658–3678.

implies that $S_g = S_{t(g)}$, for all $g \in \mathsf{G}$. Hence (1) is satisfied and α is group-type. For examples of group-type partial actions that are not global we refer § 3.2 of [BPP].

Remark 2.4

Assume that α is a group-type partial action of G on S and consider $\tau = \{\tau_y\}_{y \in \mathsf{G}_0}$ a transversal in G for x such that (1) *is true. Given $z \in \mathsf{G}_0$, fix the set $\gamma = \{\tau_y \tau_z^{-1}\}_{y \in \mathsf{G}_0}$. By Remark 3.4 of* [*BPP*], *γ is a transversal in G for z that also satisfies* (1). *Thus, the notion of group-type partial action does not depend on the choice of the object x.*

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Let G be a groupoid and $\mathrm{G} = \cup_{Y \in \mathrm{L}_0 /\sim} \mathrm{G}_Y$ its decomposition in disjoint connected components.

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Let G be a groupoid and $\mathsf{G} = \cup_{Y \in \mathsf{L}_0 /\sim} \mathsf{G}_Y$ its decomposition in disjoint connected components. Assume that $\alpha = (S_g, \alpha_g)_{g \in \mathsf{G}}$ is a partial action of G on a ring S and that $S = \oplus_{z \in \mathsf{G}_0} S_z$. In this case, for each $Y \in \mathsf{G}_0 /\sim$ we have a partial action α_Y of G_Y on S_Y, where

$$S_Y := \bigoplus_{y \in Y} S_y, \qquad \alpha_Y := (S_g, \alpha_g)_{g \in \mathsf{G}_Y}.$$

Definition 2.5

Let G be a groupoid and $\alpha = (S_g, \alpha_g)_{g\in \mathsf{G}}$ a partial action of G on the ring $S = \oplus_{z\in \mathsf{G}_0} S_z$. We say that α is group-type if the partial action α_Y of the connected groupoid G_Y on S_Y is group-type (in the sense of Definition 2.3), for all $Y \in \mathsf{G}_0/\!\sim$.

Galois Theory

A Galois correspondence for **global** actions was given in

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Next step: Galois correspondence for **partial** actions.

Let G be a finite groupoid and let $\alpha = (S_g, \alpha_g)_{g \in \mathsf{G}}$ be a partial action of G on a commutative ring S. We will assume $S_g = S1_g$, where 1_g is a central idempotent of S, $1_g \neq 0$, for all $g \in \mathsf{G}$.

Definition 3.1

The ring extension $R \subset S$ is called an α_{G}-partial Galois extension if $R = S^{\alpha_{\mathsf{G}}} := \{s \in S : \alpha_g(s1_{g^{-1}}) = s1_g, \text{ for all } g \in \mathsf{G}\}$ and there exist a positive integer m and elements $a_i, b_i \in S$, $1 \leq i \leq m$, such that

$$\sum_{1\leq i\leq m} a_i\alpha_g(b_i 1_{g^{-1}}) = \sum_{z\in\mathsf{G}_0} \delta_{z,g}1_z, \qquad \text{for all } g \in \mathsf{G}\,. \tag{2}$$

The set $\{a_i, b_i\}_{1\leq i\leq m}$ is called a partial Galois coordinate system of S over R.

Proposition 3.2

Let $R = S^{\alpha_G}$ and $R_j = S_j^{\alpha_j}$, for each $1 \leq j \leq r$. Then $R \subset S$ is an α_G-partial Galois extension if and only if $R_j \subset S_j$ is an α_j-partial Galois extension, for all $1 \leq j \leq r$.

Definition 3.3

The set

$$S^{\alpha_G} := \{a \in S : \alpha_g(a1_{g^{-1}}) = a1_g, \ \textit{for all } g \in G\}$$

is called subring of invariant elements of S.

Given a subring T of S, we will denote by G_T the set of elements of G which act trivially on T, that is,

$$\mathsf{G}_T := \{g \in \mathsf{G} \,:\, \alpha_g(t1_{g^{-1}}) = t1_g, \text{ for all } t \in T\}.$$

Proposition 3.4

Let $T = S^{\alpha_H}$, $T_{y_j} = S_{y_j}^{\alpha_{H_j(y_j)}}$ *for all* $j = 1, \ldots, r$ *and* $g \in \mathsf{G}$. *The following assertions are satisfied:*

(i) G_T *is a wide subgroupoid of* G *if and only if* $\mathsf{G}(y_j)_{T_{y_j}}$ *is a subgroup of* $\mathsf{G}(y_j)$*, for all* $1 \leq j \leq r$*,*

(ii) $\mathsf{G}_T = H$ *if and only if* $\mathsf{G}(y_j)_{T_{y_j}} = H_j(y_j)$*, for all* $1 \leq j \leq r$*.*

We recall that a unital ring extension $R \subset T$ is called *separable* if the multiplication map $m : T \otimes_R T \to T$ is a splitting epimorphism of T-bimodules. This is equivalent to saying that there exists an element $e \in T \otimes_R T$ such that $te = et$, for all $t \in T$, and $m(e) = 1_T$. Such an element e is usually called *an idempotent of separability* of T over R.

Let

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and

$$\mathfrak{L}_w(\mathsf{G}) := \{H \in \mathfrak{L}(\mathsf{G}) : H_0 = \mathsf{G}_0\}.$$

Lemma 3.5

Let $H \in \mathfrak{L}_w(\mathrm{G})$, $T = S^{\alpha_H}$ and $R = S^{\alpha_G}$. For each $1 \leq j \leq r$, consider $T_{y_j} = S_{y_j}^{\alpha_{H_j(y_j)}}$ and $R_{y_j} = S_{y_j}^{\alpha_{\mathrm{G}(y_j)}}$. Then the following statements are equivalent:

(i) *$R \subset T$ is separable,*

(ii) *$R_{y_j} \subset T_{y_j}$ is separable, for all $1 \leq j \leq r$.*

Definition 3.6

Let T be a subring of S and $T_y := T1_y$, for all $y \in \mathsf{G}_0$.

(i) *The subring T_y of S_y will be called $\alpha_{\mathsf{G}(y,z)}$-strong if for any $g, h \in \mathsf{G}(y,z)$ such that $g^{-1}h \notin \mathsf{G}(y)_{T_y}$ and for any non-zero idempotent $e \in S_g \cup S_h$, there exists $t_y \in T_y$ such that $\alpha_g(t_y 1_{g^{-1}})e \neq \alpha_h(t_y 1_{h^{-1}})$.*

(ii) *We shall say that T is α-strong if T_y is $\alpha_{\mathsf{G}(y,z)}$-strong for all $y, z \in \mathsf{G}_0$.*

Proposition 3.7

Let $H \in \mathfrak{L}_w(\mathsf{G})$, $T = S^{\alpha_H}$ and $T_{y_j} = S_{y_j}^{\alpha_{H_j(y_j)}}$ for all $1 \leq j \leq r$. The following statements are equivalent:

(i) *T is α-strong,*

(ii) *for any $g, h \in \mathsf{G}$ such that $t(g) = t(h)$ and $g^{-1}h \notin \mathsf{G}_T$ and for any non-zero idempotent $e \in S_g \cup S_h$, there exists $t \in T$ such that $\alpha_g(t1_{g^{-1}})e \neq \alpha_h(t1_{h^{-1}})e$,*

(iii) *T_{y_j} is $\alpha_{\mathsf{G}(y_j)}$-strong for all $1 \leq j \leq r$.*

In the next two propositions we establish the Galois correspondence.

Proposition 3.8

Let S be an α_{G}-partial Galois extension of $R = S^{\alpha_{\mathsf{G}}}$, $H \in \mathfrak{L}_w(\mathsf{G})$ and $T = S^{\alpha_H}$. Then

(i) *T is R-separable and α-strong,*

(ii) $\mathsf{G}_T = H$.

Proposition 3.9

Let S be an α_G-partial Galois extension of $R := S^{\alpha_{\mathsf{G}}}$, T an R-separable and α-strong subring of S such that $\mathsf{G}_T = H$, where $H \in \mathfrak{L}_w(\mathsf{G})$. Then $S^{\alpha_H} = T$.

Let $R = S^{\alpha_{\mathsf{G}}}$ and denote by $\mathfrak{B}(S)$ the set of all subrings T of S which are R-separable, α-strong and such that $\mathsf{G}_T = H$, for some $H \in \mathfrak{L}_w(\mathsf{G})$.

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Theorem 3.10

(Galois Correspondence) *Let $\alpha_G = (S_g, \alpha_g)_{g \in \mathsf{G}}$ be a unital group-type partial action of a finite groupoid G on a ring S such that $S_g = S1_g$ and $1_g \neq 0$, for all $g \in \mathsf{G}$. If S is an α_{G}-partial Galois extension of $S^{\alpha_{\mathsf{G}}}$ then there exists a bijective correspondence between $\mathfrak{L}_w(\mathsf{G})$ and $\mathfrak{B}(S)$ given by $H \mapsto S^{\alpha_H}$ whose inverse is given by $T \mapsto \mathsf{G}_T$.*

The problem of determining a Galois correspondence for partial groupoid actions (not necessarily of group-type) remains an open question.

- D. Bagio, A. Paques, *Partial groupoid actions: globalization, Morita theory and Galois theory*, Comm. Algebra 40 (10) (2012), 3658–3678.
- D. Bagio, A. Paques and H. Pinedo, *On partial skew groupoids rings*, Internat. J. Algebra Comput. 31 (1) (2021), 1–17.
- D. Bagio, A. Sant'Ana, T. Tamusiunas, *Galois correspondence for group-type partial actions of groupoids*, arXiv:2108.01180v1, 2021.
- A. Paques, T. Tamusiunas, *The Galois correspondence theorem for groupoid actions*, J. Algebra 509 (2018), 105–123.