Ribbon Categories and RT Invariants

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Abstract

It is well known that the category of finite dimensional representations of the quantum group $U_q(\mathfrak{sl}_2)$ carries the structure of a ribbon category. We present a computer program which computes the associated Reshetikhin-Turaev (RT) invariant of an inputted knot.

1. Introduction

Our main goal is to present the construction of knot invariants from a quasi-triangular Hopf algebra, using the structures present in its category of finite dimensional representations. We'll present the basic definitions for these algebras and, as we go along, introduce a pictorial technique for representing morphisms in these categories in a way that we can relate to knot diagrams.

2. Hopf Algebras

An algebra over a field k can be viewed as a triple (A, μ, η) where A is a vector space over k and the product and the unit are expressed by the linear maps $\mu : A \otimes A \to A$ and $\eta : k \to A$. In this case, the associativity of the product can be expressed as the commutativity of the following diagram:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

Similarly, the unit axiom can be expressed in terms of a diagram (which we omit).
Now the definition of coalgebra is obtained by "reversing the arrows". Explicitly

Definition 1
A coalgebra over a field k is given by a triple (C, Δ, ε) where C is a vector space and $\Delta : C \to C \otimes C$ and $\varepsilon : A \to k$ are linear maps satisfying the following axioms:

(Coassoc): The following square commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"] & C \otimes C \otimes C \end{tikzcd}$$

(Coun): The following diagram commutes:

$$\begin{tikzcd} k \otimes C & C \otimes C \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\varepsilon"] & C \otimes k \\ & C \arrow[ul, "\cong"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] & \end{tikzcd}$$

Definition 2
[3, Definition III.2.2], [3, Theorem III.2.1] A bialgebra is a quintuple $(H, \mu, \eta, \Delta, \varepsilon)$ where (H, μ, η) is an algebra and (H, Δ, ε) is a coalgebra verifying the following condition:

- The maps Δ and ε are morphisms of algebras.

Now we are able to define a Hopf algebra:

Definition 3
Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. An endomorphism S of H is called an antipode for the bialgebra H if

$$S \star \mathrm{id}_H = \mathrm{id}_H \star S = \eta \circ \varepsilon$$

(where $f \star g = \mu \circ (f \otimes g) \circ \Delta$ is convolution). A Hopf algebra is a bialgebra with an antipode.

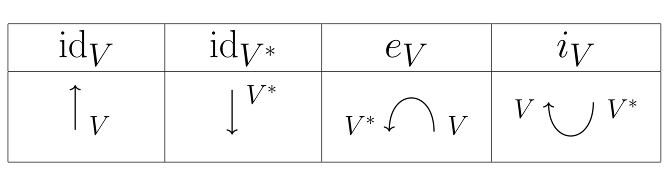
A nice thing about Hopf algebras is that their category of finite dimensional representations forms what is called a **Rigid** Monoidal Category.

Definition 4
Let $\mathcal{C}$ be a monoidal category and V be an object in $\mathcal{C}$. A *right dual* to V is an object V^* with two morphisms

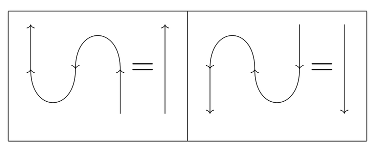
$$e_V : V^* \otimes V \to \mathbf{1},$$
$$i_V : \mathbf{1} \to V \otimes V^*$$

satisfying some compatibility conditions called *rigidity axioms.* Similarly, we can define a *left dual* object ${}^*V$ with similar morphisms and axioms. A category $\mathcal{C}$ is called *rigid* if every object in $\mathcal{C}$ has right and left duals.

In our pictorial notation, the morphisms $\mathrm{id}_V, \mathrm{id}_{V^*}, e_V$ and i_V will be represented by

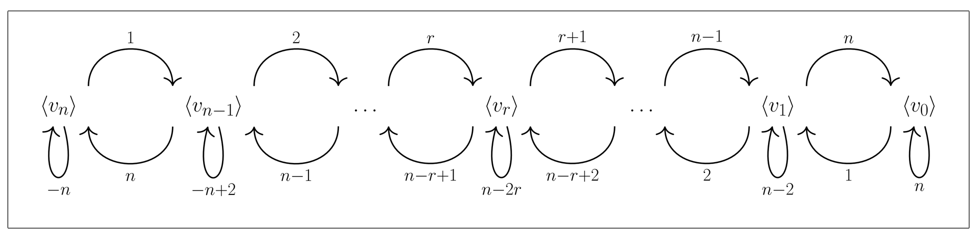


so that the rigidity conditions can be expressed by

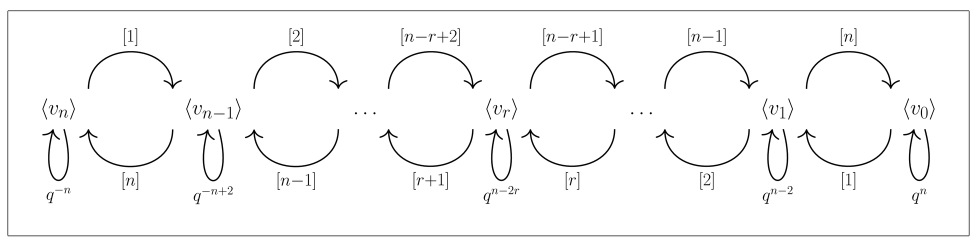


3. Quantum Groups

Quantized enveloping algebras, or quantum groups, form a famous class of Hopf algebras. In a sense they are deformations of universal enveloping algebras of finite dimensional simple Lie algebras.
The finite dimensional irreducible representation V_n of $\mathfrak{sl}_2$ of dimension $n+1$ can be represented by the picture below, where the actions of E, F and H are given, respectively, by the arrows placed above, below, and the loops.



There is a corresponding representation V_n of $U_q(\mathfrak{sl}_2)$, given by the picture below, where the actions of E, F and q^H are given, respectively, by the arrows placed above, below, and the loops.



Here

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-n+1} + q^{-n+3} + \cdots + q^{n-3} + q^{n-1}.$$

The commutation relations among the operators $E, F, q^H \in U_q(\mathfrak{sl}_2)$ reduce to those among the operators E, F, H in the limit $q \to 1$ (Serre relations, etc.). The formal definition of $U_q(\mathfrak{sl}_2)$ is given in terms of these deformed relations [2, ch. 6].
It turns out that $U_q(\mathfrak{sl}_2)$ (and $U_q(\mathfrak{g})$ in general) carries the structure of a Hopf algebra with, in particular

$$\Delta(E) = E \otimes q^H + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + q^{-H} \otimes F.$$

4. Quasitriangular Hopf Algebras and Ribbon categories

The algebras $U_q(\mathfrak{g})$ are more than just Hopf algebras, they are quasitriangular.

Definition 5
A Hopf algebra, H, is quasitriangular if there exists an invertible element R of $H \otimes H$ such that

- $(T \circ \Delta)(x) = R\Delta(x)R^{-1}, \forall x \in H,$
- $(\Delta \otimes 1)(R) = R_{13}\, R_{23},$
- $(1 \otimes \Delta)(R) = R_{13}\, R_{12},$

where

- Δ is the comultiplication on H
- $T : H \otimes H \to H \otimes H$ is the linear map given by

$$T(x \otimes y) = y \otimes x$$

- $R_{12} = \phi_{12}(R)$, where $\phi_{12} : H \otimes H \to H \otimes H \otimes H$ is the algebra morphism determined by $\phi_{12}(a \otimes b) = a \otimes b \otimes 1$ and similarly for R_{13} and R_{23}.

R is called the R-matrix.

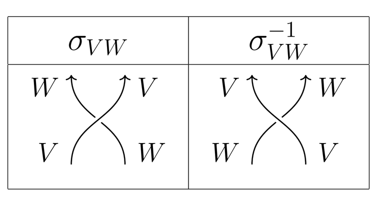
A great thing for us is that, for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$, we have a nice formula for an R-matrix on $U_q(\mathfrak{g})$.

Theorem 6
[2, Proposition 6.4.8] For $\mathfrak{g} = \mathfrak{sl}_2$, and U and V representations,

$$R = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}(1 - q^{-2})^n}{[n]!} q^{\frac{1}{2}h\otimes h} e^n \otimes f^n$$

is an R-matrix. Here $q^{h\otimes h}(u \otimes v) = q^{\lambda\mu} u \otimes v$ for $u \in U_\lambda$ and $v \in V_\mu$.

As a consequence of the existence of an R-matrix for H, its category $\mathcal{C}$ of representations is braided, i.e., for any pair of objects V, W of $\mathcal{C}$, there are functional isomorphisms $\sigma_{VW} : V \otimes W \to W \otimes V$. They are pictorially represented as follows:



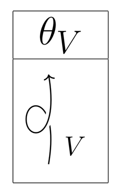
The category of finite dimensional $U_q(\mathfrak{g})$-modules has one additional structure: it is a **ribbon category**.

Definition 7
A *ribbon category* is a rigid braided tensor category with functorial isomorphisms $\delta_V : V \xrightarrow{\sim} V^{**}$ satisfying:

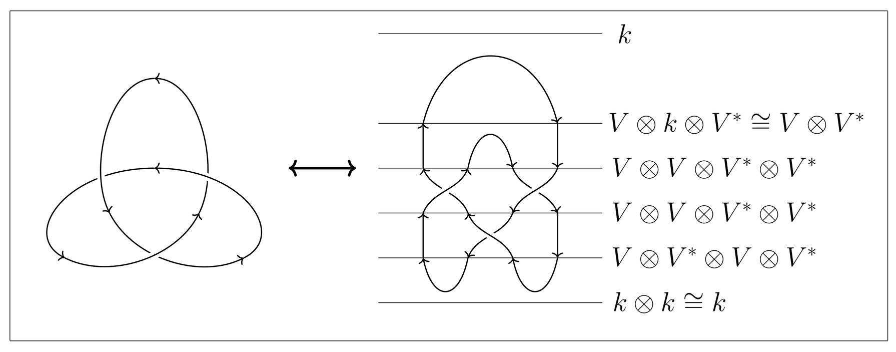
1. $\delta_{V\otimes W} = \delta_V \otimes \delta_W$
2. $\delta_{\mathbf{1}} = \mathrm{id}$
3. $\delta_{V^*} = (\delta_V^*)^{-1}$

Here, δ_V^* is the image of δ_V by a isomorphism Hom$(U, V) \cong$ Hom(V^*, U^*), as seen in [1, Lemma 2.1.6].

By [1, ch. 2], in a ribbon category we can construct functorial isomorphisms $\psi_V : V^{**} \to V$ in a way that $\theta_V = \psi_V \circ \delta_V$ satisfy some desired axioms called *balancing axioms.* The pictorial representation of θ_V is



Fixing an oriented knot diagram and an object V of a ribbon category $\mathcal{C}$, and using the pictures introduced above as building blocks, we can produce a morphism $\mathbf{1} \to \mathbf{1}$ in $\mathcal{C}$, as illustrated below in the case of the trefoil knot



For us $\mathcal{C}$ is the category of representations of the quantum group $U_q(\mathfrak{g})$. Thus $\mathbf{1} = k$ and the morphism $k \to k$ amounts to an element of k. If this number can be shown to be invariant under ambient isotopy, then it is a knot invariant.
There is a problem. In most examples $\theta_V \neq \mathrm{id}_V$, while the braids corresponding to θ_V and id_V are obviously isotopic. The solution to this problem is replace the strands of the tangle by *ribbons*, which are homeomorphic images of rectangles in $\mathbb{R}^3$. So, proceeding as in [1, Chapter 2], we consider the strands introduced above as ribbons facing upwards, and θ_V is interpreted as a ribbon that twists around itself once. The key result is now:

Theorem 8
[1, Theorem 2.3.9] (Reshetikhin-Turaev) The morphism φ depends only on the isotopy class of the tangle $F(\varphi)$, i.e., if $F(\varphi_1)$ and $F(\varphi_2)$ are isotopic as ribbon tangles then $\varphi_1 = \varphi_2$.

The assignment of a number to a knot or link as above, is called the Reshetikhin-Turaev (RT) invariant, colored by V.

5. A `Python` program

We wrote a program in `Python` implementing RT invariants for $U_q(\mathfrak{sl}_2)$. The basic input parameter is a positive integer n equal to the dimension of the irreducible $U_q(\mathfrak{sl}_2)$-representation used to label the strands of the knot diagram. Explicit matrix forms of operators E, q^H, F, and the R-matrix, are computed.
The knot diagram is introduced via a simple text interface. For example, $e_V : V^* \otimes V \to k$, corresponding to the diagram above, is represented by the string `'<n'`.
A sample input representing the trivial knot is as follows `[['<u'], ['n>']]`. The trefoil is input as `[['n>'], ['I', 'n>', 'I'], ['R^^', 'Rvv'], ['I', 'Rv^under', 'I'], ['<u', '<u']]`. This yields the invariant

$$q^{9/2}(q^{-2} + q^{-6} - q^{-8})$$

(using the $U_q(\mathfrak{sl}_2)$-module V_1 as label). It's known that the Jones polynomial of the trefoil is

$$-Q^{-4} + Q^{-3} + Q^{-1},$$

which we see coincides with the V_1-invariant above, up to a scalar factor, where $Q = q^2$. In general the Jones polynomial is more-or-less the invariant associated with $U_q(\mathfrak{sl}_2)$ and its representation V_1. The RT invariants associated with $U_q(sl_2)$ and representations V_2, V_3, V_4 etc., are similarly related to the *colored Jones polynomials.*

Acknowledgments

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