Induced maps of the Galois map for groupoid actions

Juliana Borges Pedrotti, Advisor: Thaísa Raupp Tamusiunas

Programa de Pós-Graduação em Matemática - Universidade Federal do Rio Grande do Sul

Abstract

Any action of a groupoid on a ring (not necessarily commutative) gives rise to a natural map from the set of the subgroupoids into the set of subrings, which we call the Galois map for groupoid actions. In this work we will introduce some induced maps of the Galois map and study relations between them. Furthermore, we give some conditions for the Galois map for groupoid actions to be injective.

1. Introduction

Given a finite groupoid $\mathcal{G}$ and a unital ring R, any action of $\mathcal{G}$ on R gives rise to a natural map from the set of the subgroupoids of $\mathcal{G}$ into the set of subrings of R, where each subgroupoid is taken into the fixed part by its action. This map is an inverting-inclusion map, in the sense that the subring R^β of all the invariants of R by β is a subring of $R^{\beta_\mathcal{H}}$, for any subgroupoid $\mathcal{H}$ of $\mathcal{G}$. Moreover, in the case that R is an extension of R^β, this map is called the Galois map. In this work, we will denote it by $\theta : \mathcal{H} \mapsto R^{\beta_\mathcal{H}}$.

In the commutative context, there exists a bijective correspondence between the wide subgroupoids of $\mathcal{G}$ and the R^β-subalgebras of R that are separable and β-strong proved by Paques and Tamusiunas in Ref. [3].

For group actions but in the noncommutative context, Szeto and Xue give some conditions in Ref. [5] for the Galois map to be injective. The authors also define two maps induced by θ and study relations between them.

In this work we shall extend the results of Ref. [5] to the groupoid context. We introduce two maps induced by the Galois map, $\sigma : \mathcal{H} \mapsto \theta(\mathcal{H})C(R)$ and $\gamma : \mathcal{H} \mapsto V_R(\theta(\mathcal{H}))$, and another two maps, induced by σ and γ, $\overline{\sigma} : \overline{\mathcal{H}} \mapsto \theta(\mathcal{H})C(R)$ and $\overline{\gamma} : \overline{\mathcal{H}} \mapsto V_R(\theta(\mathcal{H}))$. If R is a β-Galois extension of R^β such that R^β is $C(R)^\beta$-separable, we have that $\overline{\sigma}$ and $\overline{\gamma}$ are injective, from the set $\{\overline{\mathcal{H}} \mid \mathcal{H} \text{ is a subgroupoid of } \mathcal{G}\}$ to the set $\{T \mid T \text{ is } C(R)\text{-separable subalgebra of } R\}$, where $C(R)$ is the center of R. Finally, we prove relations between σ, γ and θ, and give some conditions for θ to be injective.

2. Definitions and notations

Groupoids are usually presented as small categories whose morphisms are all invertible. However, in this work will be adopted the algebraic definition of a groupoid, which appears, for instance, in [3].

A groupoid is a nonempty set $\mathcal{G}$, equipped with a partially defined binary operation (which we will denote by concatenation) that satisfies the following conditions:

(i) for all $g, h, l \in \mathcal{G}$, $g(hl)$ exists if and only if $(gh)l$ exists and in this case they are equal;

(ii) for all $g, h, l \in \mathcal{G}$, $g(hl)$ exists if and only if gh and hl exist;

(iii) for each $g \in \mathcal{G}$, there exist (unique) identities $d(g), r(g) \in \mathcal{G}$ such that $gd(g)$ and $r(g)g$ exist and $gd(g) = g = r(g)g$;

(iv) for each $g \in \mathcal{G}$ there exists $g^{-1} \in \mathcal{G}$ such that $d(g) = g^{-1}g$ and $r(g) = gg^{-1}$.

For all $g, h \in \mathcal{G}$, we write $\exists gh$ whenever the product gh is defined. We will denote by $\mathcal{G}^2$ the subset of the pairs $(g, h) \in \mathcal{G} \times \mathcal{G}$ such that $d(g) = r(h)$. The elements $d(g)$ and $r(g)$ are called domain and range of g, respectively.

An element $e \in \mathcal{G}$ is called an identity of $\mathcal{G}$ if $e = d(g) = r(g^{-1})$, for some $g \in \mathcal{G}$. It will be denoted by $\mathcal{G}_0$ the set of all identities of $\mathcal{G}$ and by $\mathcal{G}_e$ the set of all $g \in G$ such that $d(g) = r(g) = e$.

Given a groupoid $\mathcal{G}$ and a nonempty subset $\mathcal{H}$ of $\mathcal{G}$, we say that $\mathcal{H}$ is a subgroupoid of $\mathcal{G}$ if $\mathcal{H}$ equipped with the restriction of the operation of $\mathcal{G}$ is a groupoid itself. We say that $\mathcal{H}$ is wide if $\mathcal{H}_0 = \mathcal{G}_0$.

Consider R an algebra with 1 over a commutative ring A. According to [1], an action of $\mathcal{G}$ over R is a pair

$$\beta = (\{E_g\}_{g\in\mathcal{G}}, \{\beta_g\}_{g\in\mathcal{G}})$$

where for each $g \in G$, $E_g = E_{r(g)}$ is an ideal of R and $\beta_g : E_{g^{-1}} \longrightarrow E_g$ is an isomorphism of A-algebras satisfying the following conditions:

(i) β_e is the identity map Id_{E_e} of E_e for all $e \in \mathcal{G}_0$;

(ii) $\beta_g(\beta_h(r)) = \beta_{gh}(r)$, for all $(g, h) \in \mathcal{G}^2$ and for all $r \in E_{h^{-1}} = E_{(gh)^{-1}}$.

In this text we will consider actions of $\mathcal{G}$ on R such that each E_g is a unital A-algebra. Denote by 1_e the identity element of E_e.

Let S be any subalgebra of R and $\beta = (\{E_g\}_{g\in\mathcal{G}}, \{\beta_g\}_{g\in\mathcal{G}})$ an action of a groupoid $\mathcal{G}$ on an algebra R. Consider the set $\mathcal{H}_S = \{g \in \mathcal{G} \mid \beta_g(s1_{g^{-1}}) = s1_g, \forall s \in S\}$. From Lemma 2.1 [3], we know that $\mathcal{H}_S$ is a subgroupoid of $\mathcal{G}$.

3. Galois theory for groupoid actions

In this section we will present some results for the Galois theory for groupoid actions which were introduced in the literature in Ref. [1, 3, 4].

The subalgebra of R of the elements which are invariant under β we will denote by

$$R^\beta = \{r \in R \mid \beta_g(r1_{g^{-1}}) = r1_g \forall g \in \mathcal{G}\}.$$

We say that R is a β-Galois extension of R^β if there exist elements $x_i, y_i \in R, 1 \le i \le n$ such that $\sum_{i=1}^n x_i\beta_g(y_i1_{g^{-1}}) = \delta_{1,g}1_e$ for all $g \in G$ and $e \in G_0$.

An A-algebra R is separable over A (or A-separable) if R is a left projective $R \otimes_A R^o$-module, which is equivalent to say that there exists an element $e = \sum_{i=1}^n x_i \otimes y_i \in R \otimes_A R^o$ such that $\sum_{i=1}^n x_iy_i = 1$ and $\sum_{i=1}^n rx_i \otimes y_i = \sum_{i=1}^n x_i \otimes ry_i$ for all $r \in R$. Every $R|_{R^\beta}$ Galois extension is R^β-separable.

Let S be any subring of R. The set $V_R(S) = \{r \in R \mid rs = sr, \text{ for all } s \in S\}$ is a subring of R, called the commutator of S in R. We also define $J_g = \{r \in E_g \mid r\beta_g(x1_{g^{-1}}) = xr, \text{ for all } x \in R\}$, for $g \in G$. The J_g's describe the commutator of R^β in R, which is shown in Lemma 2.

From now on, along all the text, A is a commutative ring, $\mathcal{G}$ is a finite groupoid, $\beta = (\{E_g\}_{g\in\mathcal{G}}, \{\beta_g\}_{g\in\mathcal{G}})$ an action of $\mathcal{G}$ on R such that E_g is a unital A-algebra and $R = \oplus_{e\in G_0}E_e$. In this case, $1_R = \sum_{e\in G_0} 1_e$.

Proposition 1

[[4], Proposition 2.2] Let R be a β-Galois extension of R^β, $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ and $R_\mathcal{H} = \oplus_{e\in\mathcal{H}_0}E_e$. Then, $\beta_\mathcal{H} = \{\beta_h : E_{h^{-1}} \longrightarrow E_h \mid h \in \mathcal{H}\}$ is an action of $\mathcal{H}$ on $R_\mathcal{H}$ and $R_\mathcal{H}$ is a $\beta_\mathcal{H}$-Galois extension of $(R_\mathcal{H})^{\beta_\mathcal{H}}$.

We say that R is a β-Galois algebra of R^β if R is a β-Galois extension of R^β such that R^β is contained in the center $C(R)$ of R, and R is called a central β-Galois algebra if R is a β-Galois extension of its center $C(R)$. In particular, R is an Azumaya algebra if it is a separable extension of its center ([2].)

4. Induced Maps

By keeping definitions and notations given in Sections 2 and 3, in this section we assume that R is a β-Galois algebra of R^β such that R^β is $C(R)^\beta$-separable and we fix the notation $\theta : \mathcal{H} \mapsto R^{\beta_\mathcal{H}}$ for the Galois map from the set of the subgroupoids of $\mathcal{G}$ into the set of the R^β-subalgebras of R. For a subgroupoid $\mathcal{H}$ of $\mathcal{G}$, let $S_\mathcal{H} = \{g \in \mathcal{H} \mid J_g \neq \{0\}\}$, $T_\mathcal{H} = \{g \in \mathcal{H} \mid J_g = \{0\}\}$ and $\overline{\mathcal{H}} = \{\mathcal{L} \mid \mathcal{L} \text{ is a subgroupoid of } \mathcal{G} \text{ and } S_\mathcal{L} = S_\mathcal{H}\}$. We define two maps induced by the Galois map: $\sigma : \mathcal{H} \mapsto \theta(\mathcal{H})C(R)$ and $\gamma : \mathcal{H} \mapsto V_R(\theta(\mathcal{H}))$.

For this, we will list and prove some results.

Lemma 2

[[4], Lemma 3.1] $V_R(R^\beta) = \bigoplus_{g\in\mathcal{G}} J_g$.

Lemma 3

Let R be a β-Galois extension of R^β such that R^β is $C(R)^\beta$-separable. Then $R^{\beta_\mathcal{H}}$ is $C(R)^\beta$-separable, for each $\mathcal{H}$ subgroupoid of $\mathcal{G}$ such that $\mathcal{G}_0 \subseteq \mathcal{H}$, where $\beta_\mathcal{H} = \{\beta_h : E_{h^{-1}} \longrightarrow E_h \mid h \in \mathcal{H}\}$ is an action of $\mathcal{H}$ over $R_\mathcal{H} = \bigoplus_{e\in\mathcal{H}_0} E_e$.

Lemma 4

Let R be a β-Galois extension of R^β such that R^β is $C(R)^\beta$-separable. Then, $\overline{\gamma} : \overline{\mathcal{H}} \mapsto V_R(\theta(\mathcal{H}))$ for a subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is well defined.

Lemma 5

Let R be a β-Galois extension of R^β such that R^β is $C(R)^\beta$-separable. Then, $\overline{\sigma} : \overline{\mathcal{H}} \mapsto \theta(\mathcal{H})C(R)$ for a subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is well defined.

Lemma 6

Let R be a β-Galois extension of R^β and $\phi : S \longrightarrow \bigoplus \sum_{g\in S} J_g$ for $S \subseteq S_G$. Then, ϕ is a injective map.

Theorem 7

Let R be a β-Galois extension of R^β such that R^β is $C(R)^\beta$-separable. Then, $\overline{\sigma} : \overline{\mathcal{H}} \mapsto \theta(\mathcal{H})C(R)$ and $\overline{\gamma} : \overline{\mathcal{H}} \mapsto V_R(\theta(\mathcal{H}))$ are injective.

Proof. Let $\overline{\sigma}(\overline{\mathcal{H}}) = \overline{\sigma}(\overline{\mathcal{L}})$. Note that,

$$R^{\beta_\mathcal{H}}C(R) = \theta(\mathcal{H})C(R) = \overline{\sigma}(\overline{\mathcal{H}}) = \overline{\sigma}(\overline{\mathcal{L}}) = \theta(\mathcal{L})C(R) = R^{\beta_\mathcal{L}}C(R)$$

for subgroupoids $\mathcal{H}, \mathcal{L}$ of $\mathcal{G}$. Then, $V_R(R^{\beta_\mathcal{H}}C(R)) = V_R(R^{\beta_\mathcal{L}}C(R))$. Moreover, $V_R(R^{\beta_\mathcal{H}}) = V_R(R^{\beta_\mathcal{H}}C(R))$ and $V_R(R^{\beta_\mathcal{L}}) = V_R(R^{\beta_\mathcal{L}}C(R))$. Thus,

$$V_R(R^{\beta_\mathcal{H}}) = V_R(R^{\beta_\mathcal{H}}C(R)) = V_R(R^{\beta_\mathcal{L}}C(R)) = V_R(R^{\beta_\mathcal{L}}).$$

From Lemma 2, $\bigoplus \sum_{h\in S_\mathcal{H}} J_h = \bigoplus \sum_{l\in S_\mathcal{L}} J_l$ and $S_\mathcal{H} = S_\mathcal{L}$ by Lemma 6. Therefore $\overline{\mathcal{H}} = \overline{\mathcal{L}}$ and this implies that $\overline{\sigma}$ is injective.

Now let $\overline{\gamma}(\overline{\mathcal{H}}) = \overline{\gamma}(\overline{\mathcal{L}})$. Then, $V_R(\theta(\mathcal{H})) = V_R(\theta(\mathcal{L}))$. By Lemma 2,

$$\bigoplus \sum_{h\in S_\mathcal{H}} J_h = V_R(R^{\beta_\mathcal{H}}) = V_R(\theta(\mathcal{H})) = V_R(\theta(\mathcal{L}))$$
$$= V_R(R^{\beta_\mathcal{L}}) = \bigoplus \sum_{l\in S_\mathcal{L}} J_l.$$

From Lemma 6, $S_\mathcal{H} = S_\mathcal{L}$. Therefore $\overline{\mathcal{H}} = \overline{\mathcal{L}}$ and this implies that $\overline{\gamma}$ is injective. □

5. Injectivity of the Galois map

Keeping the same notations given in Section 4, in this section we give some conditions for a Galois maps $\theta : \mathcal{H} \longrightarrow R^{\beta_\mathcal{H}}$ to be injective. We begin with a relation between σ and γ.

Lemma 8

$\sigma : \mathcal{H} \mapsto \theta(\mathcal{H})C(R)$ is injective if and only if $\gamma : \mathcal{H} \mapsto V_R(\theta(\mathcal{H}))$ is injective.

Lemma 9

If $\sigma : \mathcal{H} \mapsto \theta(\mathcal{H})C(R)$ or $\gamma : \mathcal{H} \mapsto V_R(\theta(\mathcal{H}))$ are injective, then the Galois map $\theta : \mathcal{H} \mapsto R^{\beta_H}$ is injective.

Proof. We assume that σ is injective. Let $\mathcal{H}$ and $\mathcal{L}$ be subgroupoids of $\mathcal{G}$ such that $\theta(\mathcal{H}) = \theta(\mathcal{L})$. Thus,

$$R^{\beta_H} = R^{\beta_L} \Rightarrow R^{\beta_\mathcal{H}}C(R) = R^{\beta_\mathcal{L}}C(R) \Rightarrow \sigma(\mathcal{H}) = \sigma(\mathcal{L}).$$

Since σ is injective, then $\mathcal{H} = \mathcal{L}$. Therefore θ is injective

Now assuming that γ is injective, by Lemma 8, σ is injective. Hence θ is injective. □

Theorem 10

If $\overline{\mathcal{H}} = \{\mathcal{H}\}$, for each $\mathcal{H}$ subgroupoid of $\mathcal{G}$, then θ is injective.

Proof. Let $\mathcal{H}$ and $\mathcal{L}$ subgroupoids of $\mathcal{G}$. Then,

$$R^{\beta_\mathcal{H}} = R^{\beta_\mathcal{L}} \Rightarrow R^{\beta_\mathcal{H}}C(R) = R^{\beta_\mathcal{L}}C(R).$$

From Lemma 5, $\sigma(\overline{\mathcal{H}}) = \sigma(\overline{\mathcal{L}})$. Since σ is injective by Theorem 7, $\overline{\mathcal{H}} = \overline{\mathcal{L}}$. By hypothesis $\overline{\mathcal{H}} = \{H\}$ and $\overline{\mathcal{L}} = \{\mathcal{L}\}$, so $\mathcal{H} = \mathcal{L}$. Therefore θ is injective. □

Let $\mathcal{H}$ be a subset of $\mathcal{G}$. The subgroupoid generated by the elements in $\mathcal{H}$ is a small subgroupoid of $\mathcal{G}$ containing $\mathcal{H}$, which will be denoted by $\langle\mathcal{H}\rangle$.

Theorem 11

Let $\langle S_\mathcal{H}\rangle$ be the subgroupoid of $\mathcal{G}$ generated by the elements in S_H, for a subgroupoid $\mathcal{H}$ of G. If $\langle S_\mathcal{H}\rangle = \mathcal{H}$, then θ is injective.

An immediate consequence of Theorem 11 is that if $J_g \neq \{0\}$ for each $g \in \mathcal{G}$, then θ is injective. In fact, since $J_g \neq \{0\}$ for each $g \in \mathcal{G}$, $\langle S_\mathcal{H}\rangle = S_\mathcal{H} = \mathcal{H}$ for each subgroupoid $\mathcal{H}$ of $\mathcal{G}$. Hence θ is injective.

6. Conclusion

In Section 4 we showed that $\overline{\sigma} : \overline{\mathcal{H}} \mapsto \theta(\mathcal{H})C(R)$ and $\overline{\gamma}$: $\overline{\mathcal{H}} \mapsto V_R(\theta(\mathcal{H}))$ are a injective correspondence from the set $\{\overline{\mathcal{H}} \mid \mathcal{H} \text{ is a subgroupoid of } \mathcal{G}\}$ to the set of the $C(R)$-separable subalgebras of R.

The Theorems 9 and 10 give sufficient conditions for that θ to be injective. Furthermore, the Theorem 10 holds for any β-Galois extension of R which is not necessarily a separable algebra over $C(R)^\beta$ and this theorem generalizes Theorem 3.3 in [4].

Acknowledgments

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES).

References

[1] D. Bagio and A. Paques. Partial groupoid actions: Globalization, morita theory and galois theory. *Communications in Algebra*, 40:3658–3678, 2012.

[2] F. DeMeyer and E. Ingraham. *Separable Algebras Over Commutative Rings*. Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, 1971.

[3] A. Paques and T. Tamusiunas. The Galois correspondence theorem for groupoid actions. *Journal of Algebra*, 509:105–123, 2018.

[4] A. Paques and T. Tamusiunas. On the galois map for groupoid actions. *Communications in Algebra*, 49:1037–1047, 2020.

[5] G. Szeto and L. Xue. The galois map and its induced maps. *Contemporary Ring Theory 2011*, 10-15, 2012.