Crossed modules over inverse semigroups, crossed module extensions and their cohomological interpretation

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Contents

1. Introduction
   - Crossed modules over groups and the corresponding extensions
   - The idea of the proof
   - Modules over inverse semigroups and inverse semigroup cohomology
2. Crossed modules over inverse semigroups and the third inverse semigroup cohomology group
   - Crossed S-modules
   - Crossed module extensions
   - From crossed module extensions of A by T to $H^3(T^1, A^1)$
   - From admissible crossed module extensions of A by T to $H^3_{\le}(T^1, A^1)$
   - From $H^3_{\le}(T^1, A^1)$ to crossed module extensions of A by T
3. References

Introduction

Outline

1. Introduction
   - Crossed modules over groups and the corresponding extensions
   - The idea of the proof
   - Modules over inverse semigroups and inverse semigroup cohomology
2. Crossed modules over inverse semigroups and the third inverse semigroup cohomology group
   - Crossed S-modules
   - Crossed module extensions
   - From crossed module extensions of A by T to $H^3(T^1, A^1)$
   - From admissible crossed module extensions of A by T to $H^3_{\le}(T^1, A^1)$
   - From $H^3_{\le}(T^1, A^1)$ to crossed module extensions of A by T
3. References

Crossed G-modules

- G and N groups.

Definition 1.1 (Whitehead [7] and Maclane [5])

A crossed G-module structure on N is an action of G on N together with a homomorphism $\beta : N \to G$ such that

1. $\beta(n)n' = nn'n^{-1}$ for all $n, n' \in N$;
2. $\beta(gn) = g\beta(n)g^{-1}$ for all $g \in G$ and $n \in N$.

Then N (with the G-action and β) is called a crossed G-module.

Examples of crossed G-modules

The classical examples are the following.

Example 1.2

1. Usual G-modules (N is abelian and β is trivial);
2. Normal subgroups of G ($N \trianglelefteq G$, G acts by conjugation and β is the inclusion map);
3. $G = \mathrm{Aut}\, N$ and $\beta(n)$ is the conjugation by n.

The corresponding 4-term exact sequence

- N is a crossed G-module.

Remark 1.3

It follows from Definition 1.1 (2) that $\beta(N) \trianglelefteq G$.

Definition 1.4

We define $A = \ker \beta$ and $H = G/\beta(N)$, so that the sequence

$$A \hookrightarrow N \xrightarrow{\beta} G \twoheadrightarrow H$$

is exact.

The induced H-module structure

- N is a crossed G-module;
- $A = \ker \beta$;
- $H = G/\beta(N)$.

Remark 1.5

It follows from Definition 1.1 (1) that $A \subseteq C(N)$, in particular, A is abelian. Moreover, by Definition 1.1 (2) we have $gA = A$ for all $g \in G$, so G acts on A, and $\beta(N) \trianglelefteq G$ acts trivially on A by Definition 1.1 (1).

Corollary 1.6

The abelian group A is a usual H-module under the induced action.

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Equivalent sequences and $H^3(G, A)$

- A is an abelian group;
- H is a group.

Definition 1.7

One defines equivalence of such 4-term exact sequences $A \xrightarrow{i} N \xrightarrow{\beta} G \xrightarrow{\pi} H$ and $A \xrightarrow{i'} N' \xrightarrow{\beta'} G' \xrightarrow{\pi'} H$ (I am not giving a precise definition here, since it will appear later in a more general context).

Theorem 1.8

There is a one-to-one correspondence between the equivalence classes of such exact sequences and the elements of $H^3(H, A)$.

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Outline

1. Introduction
   - Crossed modules over groups and the corresponding extensions
   - The idea of the proof
   - Modules over inverse semigroups and inverse semigroup cohomology
2. Crossed modules over inverse semigroups and the third inverse semigroup cohomology group
   - Crossed S-modules
   - Crossed module extensions
   - From crossed module extensions of A by T to $H^3(T^1, A^1)$
   - From admissible crossed module extensions of A by T to $H^3_{\le}(T^1, A^1)$
   - From $H^3_{\le}(T^1, A^1)$ to crossed module extensions of A by T
3. References

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The easy part

Proposition 1.9

Given $A \xrightarrow{i} N \xrightarrow{\beta} G \xrightarrow{\pi} H$, a choice of transversal of π determines a cocycle $c \in Z^3(H, A)$ up to a coboundary. Equivalent sequences induce the same element of $H^3(H, A)$.

The difficult part

- A is an H-module;
- $c \in Z^3(H, A)$.

Definition 1.10

Let G be the free group over $[H] := \{[h] \mid h \in H\}$ and $\varphi : G \to H$ the homomorphism sending $[h]$ to h. Denote $R = \ker \varphi$ and $N = A \times R$.

Proposition 1.11

The group R is freely generated by $[h][k][hk]^{-1}$, $h, k \in H$.

The difficult part

Definition 1.12

Let $r(h,k) := [h][k][hk]^{-1}$ for $h,k \in H \setminus \{1_H\}$, and $r(h,k) := 1_G$ otherwise.

Definition 1.13

Define a map $[H] \to \mathrm{Aut}\,(N)$ by $[x](a,1_G) = (xa,1_G)$ and $[x](1_A, r(y,z)) = (c(x,y,z), r(x,y)r(xy,z)r(x,yz)^{-1})$, where the action of x on a comes from the H-module structure on A. It extends to a homomorphism $G \to \mathrm{Aut}\,(N)$.

The difficult part

Proposition 1.14

The map $G \to \mathrm{Aut}(N)$ *together with* $\beta : N \to G$, $\beta(a,b) = b$, *is a crossed* G*-module structure on* N.

Proposition 1.15

Cohomologous $c, c' \in Z^3(H, A)$ *induce equivalent* $A \xrightarrow{i} N \xrightarrow{\beta} G \xrightarrow{\pi} H$ *and* $A \xrightarrow{i'} N' \xrightarrow{\beta'} G' \xrightarrow{\pi} H$.

Outline

1. Introduction
   - Crossed modules over groups and the corresponding extensions
   - The idea of the proof
   - Modules over inverse semigroups and inverse semigroup cohomology
2. Crossed modules over inverse semigroups and the third inverse semigroup cohomology group
   - Crossed S-modules
   - Crossed module extensions
   - From crossed module extensions of A by T to $H^3(T^1, A^1)$
   - From admissible crossed module extensions of A by T to $H^3_{\le}(T^1, A^1)$
   - From $H^3_{\le}(T^1, A^1)$ to crossed module extensions of A by T
3. References

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S-modules

- S an inverse semigroup;
- A a semilattice of abelian groups (i.e. a commutative inverse semigroup);

Definition 1.16 (Lausch [4])

An S-module structure on A is a pair (α, λ), where α is an isomorphism $E(S) \to E(A)$ and λ is a homomorphism $S \to \mathrm{End}(A)$ such that

1. $\lambda_e(a) = \alpha(e)a$, for all $e \in E(S)$, $a \in A$;
2. $\lambda_s(\alpha(e)) = \alpha(ses^{-1})$, for all $s \in S$, $e \in E(S)$.

The groups $C^n(S^1, A^1)$

- A an S-module.

Definition 1.17

Denote by $C^n(S^1, A^1)$ the abelian group of functions

$$\left\{ f : S^n \to A \mid f(s_1, \dots, s_n) \in A_{\alpha(s_1 \dots s_n s_n^{-1} \dots s_1^{-1})} \right\}$$

under the coordinate-wise multiplication

The cochain complex $C^n(S^1, A^1)$

Proposition 1.18

The groups $C^n(S^1, A^1)$ form a cochain complex

$$C^1(S^1, A^1) \xrightarrow{\delta^1} \dots \xrightarrow{\delta^{n-1}} C^n(S^1, A^1) \xrightarrow{\delta^n} \dots$$

under the coboundary homomorphism $\delta^n : C^n(S^1, A^1) \to C^{n+1}(S^1, A^1)$ mapping $f \in C^n(S^1, A^1)$ to $\delta^n f \in C^{n+1}(S^1, A^1)$, where

$$\begin{aligned}(\delta^n f)(s_1, \dots, s_{n+1}) = \lambda_{s_1}(f(s_2, \dots, s_{n+1})) \\ \prod_{i=1}^{n} f(s_1, \dots, s_i s_{i+1}, \dots, s_{n+1})^{(-1)^i} \\ f(s_1, \dots, s_n)^{(-1)^{n+1}}.\end{aligned}$$

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The groups $H^n(S^1, A^1)$

Definition 1.19

Denote $\ker \delta^n$ by $Z^n(S^1, A^1)$, $\operatorname{im} \delta^{n-1}$ by $B^n(S^1, A^1)$ and $Z^n(S^1, A^1)/B^n(S^1, A^1)$ by $H^n(S^1, A^1)$. The elements of $C^n(S^1, A^1)$, $Z^n(S^1, A^1)$, $B^n(S^1, A^1)$ and $H^n(S^1, A^1)$ will be called *n-cochains*, *n-cocycles*, *n-coboundaries* and *n-cohomologies* of S with values in A, respectively.

Proposition 1.20 (Dokuchaev and Khrypchenko [2])

The group $H^n(S^1, A^1)$ is isomorphic to the Lausch cohomology group $H^n(S, A)$ for all $n \geq 2$.

The groups $C^n_{\leq}(S^1, A^1)$

- $f \in C^n(S^1, A^1)$.

Definition 1.21

The n-cochain $f \in C^n(S^1, A^1)$ is said to be order-preserving, if

$$s_1 \leq t_1, \dots, s_n \leq t_n \Rightarrow f(s_1, \dots, s_n) \leq f(t_1, \dots, t_n).$$

Such n-cochains form a subgroup of $C^n(S^1, A^1)$, denoted by $C^n_{\leq}(S^1, A^1)$.

The groups $H^n_{\leq}(S^1, A^1)$

Remark 1.22

Since $\delta^n\left(C^n_{\leq}(S^1, A^1)\right) \subseteq C^{n+1}_{\leq}(S^1, A^1)$, we obtain the cochain complex

$$C^1_{\leq}(S^1, A^1) \stackrel{\delta^1}{\to} \dots \stackrel{\delta^{n-1}}{\to} C^n_{\leq}(S^1, A^1) \stackrel{\delta^n}{\to} \dots$$

Definition 1.23

One naturally defines the groups of order-preserving n-cocycles $Z^n_{\leq}(S^1, A^1)$, n-coboundaries $B^n_{\leq}(S^1, A^1)$ and n-cohomologies $H^n_{\leq}(S^1, A^1)$ of S with values in A.

Crossed modules over inverse semigroups and the third inverse semigroup cohomology group

Outline

1. Introduction
   - Crossed modules over groups and the corresponding extensions
   - The idea of the proof
   - Modules over inverse semigroups and inverse semigroup cohomology
2. Crossed modules over inverse semigroups and the third inverse semigroup cohomology group
   - Crossed S-modules
   - Crossed module extensions
   - From crossed module extensions of A by T to $H^3(T^1, A^1)$
   - From admissible crossed module extensions of A by T to $H^3_{\leq}(T^1, A^1)$
   - From $H^3_{\leq}(T^1, A^1)$ to crossed module extensions of A by T
3. References

Relatively invertible endomorphisms of A

- A a semilattice of groups.

Definition 2.1

An endomorphism $\varphi : A \to A$ is called relatively invertible if there exist $\bar\varphi \in \mathrm{End}(A)$ and $e_\varphi \in E(A)$ satisfying:

1. $\bar\varphi \circ \varphi(a) = e_\varphi a$ and $\varphi \circ \bar\varphi(a) = \varphi(e_\varphi)a$, for any $a \in A$;
2. e_φ is the identity of $\bar\varphi(A)$ and $\varphi(e_\varphi)$ is the identity of $\varphi(A)$.

The set of relatively invertible endomorphisms of A is denoted by $\mathrm{end}(A)$.

Proposition 2.2 (Proposition 3.4 from [1])

The set $\mathrm{end}(A)$ *is an inverse subsemigroup of* $\mathrm{End}(A)$ *isomorphic to* $\mathcal{I}_{ui}(A)$*, the semigroup of isomorphisms between principal ideals of* A*.*

Crossed S-modules

- S an inverse semigroup;
- N a semilattice of groups.

Definition 2.3 (Dokuchaev, Khrypchenko, Makuta [3])

A crossed S-module structure on N is a triple (α, λ, β), where α is an isomorphism $E(S) \to E(N)$, λ is a homomorphism $S \to \mathrm{end}(N)$ and β is an idempotent-separating homomorphism $N \to S$ such that $\beta|_{E(N)} = \alpha^{-1}$ and

1. $\lambda_e(n) = \alpha(e)n$, for all $e \in E(S)$, $n \in N$;
2. $\lambda_s(\alpha(e)) = \alpha(ses^{-1})$, for all $s \in S$, $e \in E(S)$;
3. $\lambda_{\beta(n)}(n') = nn'n^{-1}$, for all $n, n' \in N$;
4. $\beta(\lambda_s(n)) = s\beta(n)s^{-1}$, for all $s \in S$, $n \in N$.

- N a crossed S-module.

Proposition 2.4

The semigroup $A := \beta^{-1}(E(S))$ is a semilattice of (abelian) groups contained in $C(N)$.

Proposition 2.5

The collection $\mathcal{B} = \{\beta(N_e)\}_{e\in E(N)}$ is a group kernel normal system in S. Moreover, if $T := S/\rho_{\mathcal{B}}$ and $\pi := \rho_{\mathcal{B}}^{\natural} : S \to T$, then π is idempotent-separating and $\pi^{-1}(E(T)) = \beta(N)$.

Outline

1 Introduction
- Crossed modules over groups and the corresponding extensions
- The idea of the proof
- Modules over inverse semigroups and inverse semigroup cohomology

2 Crossed modules over inverse semigroups and the third inverse semigroup cohomology group
- Crossed S-modules
- Crossed module extensions
- From crossed module extensions of A by T to $H^3(T^1, A^1)$
- From admissible crossed module extensions of A by T to $H^3_{\leq}(T^1, A^1)$
- From $H^3_{\leq}(T^1, A^1)$ to crossed module extensions of A by T

3 References

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Crossed module extensions of A by T

- A a semilattice of abelian groups;
- T an inverse semigroup.

Definition 2.6

A crossed module extension of A by T is a 4-term sequence

$$A \xrightarrow{i} N \xrightarrow{\beta} S \xrightarrow{\pi} T,$$

where

1. N is a crossed S-module and β is the corresponding crossed module homomorphism;
2. i is a monomorphism and π is an idempotent-separating epimorphism;
3. $i(A) = \beta^{-1}(E(S))$ and $\beta(N) = \pi^{-1}(E(T))$.

Equivalent crossed module extensions of A by T

Definition 2.7

By the equivalence of crossed module extensions of A by T we mean the smallest equivalence relation identifying $A \xrightarrow{i} N \xrightarrow{\beta} S \xrightarrow{\pi} T$ and $A \xrightarrow{i'} N' \xrightarrow{\beta'} S' \xrightarrow{\pi'} T$, such that there are homomorphisms $\varphi_1 : N \to N'$ and $\varphi_2 : S \to S'$

1. making the following diagram commute

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, equal] & N \arrow[r, "\beta"] \arrow[d, "\varphi_1"] & S \arrow[r, "\pi"] \arrow[d, "\varphi_2"] & T \arrow[d, equal] \\ A \arrow[r, "i'"] & N' \arrow[r, "\beta'"] & S' \arrow[r, "\pi'"] & T \end{tikzcd}$$

2. satisfying $\varphi_1 \circ \lambda_s = \lambda'_{\varphi_2(s)} \circ \varphi_1$ for all $s \in S$.

The induced T-module structure on A

Proposition 2.8

Any crossed module extension of A by T induces a T-module structure on A. Moreover, equivalent crossed module extensions of A by T induce the same T-module structure on A.

Definition 2.9

A crossed module extension of a T-module A by T is a crossed module extension of A by T which induces the given T-module structure on A. Denote by $\mathcal{E}(T, A)$ the set of equivalence classes of crossed module extensions of a T-module A by T.

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Outline

1. Introduction
   - Crossed modules over groups and the corresponding extensions
   - The idea of the proof
   - Modules over inverse semigroups and inverse semigroup cohomology

2. Crossed modules over inverse semigroups and the third inverse semigroup cohomology group
   - Crossed S-modules
   - Crossed module extensions
   - From crossed module extensions of A by T to $H^3(T^1, A^1)$
   - From admissible crossed module extensions of A by T to $H^3_{\le}(T^1, A^1)$
   - From $H^3_{\le}(T^1, A^1)$ to crossed module extensions of A by T

3. References

Transversals that respect idempotents

- $\varphi : S \to T$ a homomorphism of inverse semigroups.

Definition 2.10

A map $\rho : \varphi(S) \to S$ such that $\varphi \circ \rho = \mathrm{id}_{\varphi(S)}$ will be called a transversal of φ. We say that ρ respects idempotents if $\rho(E(\varphi(S))) \subseteq E(S)$.

Remark 2.11

Since $E(\varphi(S)) = \varphi(E(S))$, one may always choose a transversal ρ of φ which respects idempotents.

The induced cocycle

- $A \xrightarrow{i} N \xrightarrow{\beta} S \xrightarrow{\pi} T$ a crossed module extension of A by T.

Lemma 2.12

The crossed module extension determines an element $c \in C^3(T^1, A^1)$.

1. Choose a transversal ρ of π.
2. There exists a unique $f : T^2 \to \beta(N)$ such that $f(x,y) \in \beta(N)_{\rho(xy)\rho(xy)^{-1}}$ and $\rho(x)\rho(y) = f(x,y)\rho(xy)$.
3. Choose $F : T^2 \to N$ such that $\beta(F(x,y)) = f(x,y)$, where $F(x,y) \in N_{\alpha(\rho(xy)\rho(xy)^{-1})}$.
4. There exists a unique $c : T^3 \to A$ such that $\lambda_{\rho(x)}(F(y,z))F(x,yz) = i(c(x,y,z))F(x,y)F(xy,z)$, where $c(x,y,z) \in A_{i^{-1}\circ\alpha(\rho(xyz)\rho(xyz)^{-1})}$.

The induced cohomology

Lemma 2.13

The cochain c is a cocycle from $Z^3(T^1, A^1)$.

Lemma 2.14

Another choices of ρ and F lead to a cocycle cohomologous to c.

Lemma 2.15

Equivalent extensions induce cohomologous cocycles.

Proposition 2.16

There is a well-defined function from $\mathcal{E}(T, A)$ to $H^3(T^1, A^1)$.

Outline

1. Introduction
   - Crossed modules over groups and the corresponding extensions
   - The idea of the proof
   - Modules over inverse semigroups and inverse semigroup cohomology
2. Crossed modules over inverse semigroups and the third inverse semigroup cohomology group
   - Crossed S-modules
   - Crossed module extensions
   - From crossed module extensions of A by T to $H^3(T^1, A^1)$
   - From admissible crossed module extensions of A by T to $H^3_{\le}(T^1, A^1)$
   - From $H^3_{\le}(T^1, A^1)$ to crossed module extensions of A by T
3. References

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Order-preserving transversals

- $\rho : \varphi(S) \to S$ a transversal of $\varphi : S \to T$.

Definition 2.17

We say that ρ is order-preserving whenever $x \le y \Rightarrow \rho(x) \le \rho(y)$ for all $x, y \in \varphi(S)$.

Remark 2.18

The following statements are equivalent:

1. ρ is order-preserving and respects idempotents;
2. $\rho(ex) = \rho(e)\rho(x)$ for all $e \in E(\varphi(S))$ and $x \in \varphi(S)$;
3. $\rho(xe) = \rho(x)\rho(e)$ for all $e \in E(\varphi(S))$ and $x \in \varphi(S)$.

Admissible crossed module extensions

- $A \xrightarrow{i} N \xrightarrow{\beta} S \xrightarrow{\pi} T$ a crossed module extension of A by T.

Definition 2.19

The crossed module extension of A by T will be called admissible if β and π possess order-preserving transversals which respect idempotents. The set of equivalence classes of admissible crossed module extensions of A by T will be denoted by $\mathcal{E}_{\leq}(T, A)$.

Corollary 2.20

There is a well-defined function from $\mathcal{E}_{\leq}(T, A)$ to $H^3_{\leq}(T^1, A^1)$.

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1. Introduction
   - Crossed modules over groups and the corresponding extensions
   - The idea of the proof
   - Modules over inverse semigroups and inverse semigroup cohomology
2. Crossed modules over inverse semigroups and the third inverse semigroup cohomology group
   - Crossed S-modules
   - Crossed module extensions
   - From crossed module extensions of A by T to $H^3(T^1, A^1)$
   - From admissible crossed module extensions of A by T to $H^3_{\le}(T^1, A^1)$
   - From $H^3_{\le}(T^1, A^1)$ to crossed module extensions of A by T
3. References

E-unitary covers

- T an inverse semigroup;
- G a group.

Definition 2.21 (McAlister-Reilly [6])

An *E-unitary cover* of T through G is an E-unitary inverse semigroup S, such that

1. the maximum group image of S is isomorphic to G;
2. there is an idempotent-separating epimorphism $\pi : S \to T$.

Proposition 2.22 (McAlister-Reilly [6])

Each inverse semigroup admits an E-unitary cover.

The E-unitary cover through a free group

- T an inverse semigroup;
- identify T with $\{[t] \mid t \in T\}$ and let $T^{-1} := \{[t]^{-1} \mid t \in T\}$;
- $\varphi : F(T \sqcup T^{-1}) \to T$ the epimorphism of semigroups such that $\varphi([t]) = t$ and $\varphi([t]^{-1}) = t^{-1}$;
- $\psi : F(T \sqcup T^{-1}) \to FG(T)$ the epimorphism of semigroups such that $\psi([t]) = [t]$ and $\psi([t]^{-1}) = [t]^{-1}$;
- $\Phi : T \to 2^{FG(T)}$, where $\Phi(t) = \psi(\varphi^{-1}(t))$.

Proposition 2.23 (McAlister-Reilly [6])

The semigroup $S := \Pi(T, FG(T), \Phi)$ is an E-unitary cover of T through $FG(T)$, where $\Pi(T, FG(T), \Phi) = \{(t, w) \in T \times FG(T) \mid w \in \Phi(t)\}$.

Irreducible words

- $w \in F(T \sqcup T^{-1})$.

Definition 2.24

The word w is irreducible if it has no subwords of the form uu^{-1}, where $u \in F(T \sqcup T^{-1})$.

Remark 2.25

Each $w \in FG(T) \setminus \{\epsilon\}$ admits a unique representation as a non-empty irreducible word $\mathrm{irr}\,(w)$ over $T \sqcup T^{-1}$. Hence, there is a well-defined map $\mathrm{irr} : FG(T) \setminus \{\epsilon\} \to F(T \sqcup T^{-1})$.

Equivalent representation of S

Definition 2.26

Define $\nu : FG(T) \to T^1$, where $T^1 = T \sqcup \{1\}$, by

$$\nu(w) = \begin{cases} \varphi(\mathrm{irr}\,(w)), & w \neq \epsilon, \\ 1, & w = \epsilon. \end{cases}$$

Proposition 2.27

We have $S = \{(t, w) \in T \times FG(T) \mid t \leq \nu(w)\}$ *and* $E(S) = \{(e, \epsilon) \mid e \in E(T)\} = E(T) \times \{\epsilon\}$.

Remark 2.28

If T is a group and $(t, w) \in S$, then $(t, w) = (\nu(w), w)$, so $S \cong FG(T)$.

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The semigroup N

- $\pi : S \to T$ the covering epimorphism, $\pi(t, w) = t$;
- $K := \pi^{-1}(E(T))$.

Remark 2.29

K is a semilattice of groups $K_e = \{(e, w) \in E(T) \times FG(T) \mid e \leq \nu(w)\}$.

- $N := \bigsqcup_{e \in E(T)} (A_e \times K_e)$ with coordinatewise multiplication.

Remark 2.30

If one writes the elements of N as triples (a, e, w), then
$N = \{(a, e, w) \in A \times E(T) \times FG(T) \mid a \in A_e \text{ and } e \leq \nu(w)\}$.

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- (θ, η) a T-module structure on A;
- $i : A \to N$, $i(a) = (a, \theta^{-1}(aa^{-1}), \epsilon)$;
- $\alpha : E(S) \to E(N)$, $\alpha(e, \epsilon) = (\theta(e), e, \epsilon)$;
- $\beta : N \to S$, $\beta(a, e, w) = (e, w)$.

Proposition 2.31

Given $c \in Z^3_{\le}(T^1, A^1)$, there exists a homomorphism $\lambda : S \to \operatorname{end} N$, such that (α, λ, β) is a crossed S-module structure on N and the sequence $A \xrightarrow{i} N \xrightarrow{\beta} S \xrightarrow{\pi} T$ is a crossed module extension of the T-module A by T.

The construction of λ

- $s = (t, u) \in S$;
- $n = (a, e, w) \in N$.

$$\lambda_s(n) = \begin{cases} (\zeta_t(w)\eta_t(a), tet^{-1}, uwu^{-1}), & u \neq \epsilon, \\ \alpha(s)n, & u = \epsilon, \end{cases}$$

where

$$\zeta_t(w) = \begin{cases} \xi_t(\mathrm{irr}\,(w)), & w \neq \epsilon, \\ \theta(tt^{-1}), & w = \epsilon. \end{cases}$$

The construction of $\xi_t : F(T \sqcup T^{-1}) \to A$

- $c \in Z^3_{\le}(T^1, A^1)$ strongly normalized;
- $t \in T$;
- $w \in F(T \sqcup T^{-1})$.

If $w = [x]^{-1} u$ for some $x \in T$ and $u \in F(T \sqcup T^{-1})^1$, then

$$\xi_t(w) := c(t, x^{-1}, x)^{-1} \xi_t([x^{-1}]u).$$

We now proceed by induction on $l(w)$ (the length of w).

The construction of $\xi_t : F(T \sqcup T^{-1}) \to A$

Base of induction. If $w = [x]$ for some $x \in T$, then

$$\xi_t(w) := \theta(txx^{-1}t^{-1}).$$

Inductive step. $l(w) > 1$ and $w = [x]u$ for some $x \in T$ and $u \in F(T \sqcup T^{-1})$.

Case 1. If $w = [x][y]v$ for some $x, y \in T$ and $v \in F(T \sqcup T^{-1})^1$, then

$$\xi_t(w) := c(t, x, y)\xi_t([xy]v).$$

Case 2. If $w = [x][y]^{-1}v$ for some $x, y \in T$ and $v \in F(T \sqcup T^{-1})^1$, then

$$\xi_t(w) := c(t, xy^{-1}, y)^{-1}\xi_t([xy^{-1}]v).$$

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From $H^3_{\leq}(T^1, A^1)$ to $\mathcal{E}(T, A)$

- $c, c' \in Z^3_{\leq}(T^1, A^1)$ cohomologous;
- $A \xrightarrow{i} N \xrightarrow{\beta} S \xrightarrow{\pi} T$ and $A \xrightarrow{i'} N' \xrightarrow{\beta'} S' \xrightarrow{\pi'} T$ the corresponding crossed module extensions of A by T.

Proposition 2.32

The crossed module extensions $A \xrightarrow{i} N \xrightarrow{\beta} S \xrightarrow{\pi} T$ and $A \xrightarrow{i'} N' \xrightarrow{\beta'} S' \xrightarrow{\pi'} T$ are equivalent.

Proposition 2.33

There is a map from $H^3_{\leq}(T^1, A^1)$ to $\mathcal{E}(T, A)$.

From $H^3_{\leq}(T^1, A^1)$ to $\mathcal{E}_{\leq}(T, A)$

- T an F-inverse monoid.

Lemma 2.34

The crossed module extension $A \xrightarrow{i} N \xrightarrow{\beta} S \xrightarrow{\pi} T$ is admissible.

Corollary 2.35

There is a map from $H^3_{\leq}(T^1, A^1)$ to $\mathcal{E}_{\leq}(T, A)$.

Theorem 2.36 (Dokuchaev, Khrypchenko, Makuta [3])

There is a bijective correspondence between $H^3_{\leq}(T^1, A^1)$ and $\mathcal{E}_{\leq}(T, A)$.

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THANK YOU!