ON GROUPOID ALGEBRAS WITH APPLICATIONS TO LEAVITT LABELLED PATH ALGEBRAS - TRAINING SESSION

GILLES G. DE CASTRO, DANIE W. VAN WYK

1. Leavitt path algebras as Steinberg algebras

We briefly recall some definitions seen during the first two talks. The exercise will be given at the end of the section. Let $E = (E^1, E^0, r, s)$ be a directed graph.

- For $x \in E^*$, we let $xE^1 = \{e \in E^1 : s(e) = r(x)\}$.
- The **boundary path space** is

$$\partial E = E^\infty \cup \{x \in E^* : xE^1 = \emptyset\} \cup \{x \in E^* : |xE^1| = \infty\}.$$

- $xE^1 = \emptyset$ means that $r(x)$ is a sink, $|xE^1| = \infty$ means that $r(x)$ is an infinite emitter.

For $\mu \in E^*$ and $F \subseteq r(\mu)E^1$ finite, define the **cylinder set**

$$Z(\mu) = \{\mu x \in \partial E : x \in r(\mu)\partial E\}$$

and the **punctured cylinder set**

$$Z(\mu \backslash F) = Z(\mu) \backslash \cup_{e \in F} Z(\mu e).$$

Define

$$G_E = \{(\alpha x, |\alpha| - |\beta|, \beta x) \in \partial E \times \mathbb{Z} \times \partial E : x \in \partial E, \alpha, \beta \in E^*\}$$

and

- multiplication by $(x, k, y)(y, l, z) = (x, k + l, z)$
- inversion by $(x, k, y)^{-1} = (y, -k, x)$.

Then

- G_E is a groupoid,
- $r(x, k, y) = (x, 0, x)$ and $s(x, k, y) = (y, 0, y)$, and
- $G_E^{(0)} = \{(x, 0, x) : x \in \partial E\}$ is identified with ∂E.

Using the cylinder sets we can define a topology on G_E: for $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$ and $F \subseteq r(\mu)E^1$ finite we define

$$Z(\mu, \nu) = \{(\mu x, |\mu| - |\nu|, \nu x) \in G_E : \mu x \in Z(\mu), \nu x \in Z(\nu)\}$$

and

$$Z(\mu, F, \nu) = \{(\mu x, |\mu| - |\nu|, \nu x) \in G_E : \mu x \in Z(\mu \backslash F), \nu x \in Z(\nu \backslash F)\}.$$

These sets form a basis of compact open bisections for a locally compact Hausdorff topology on G_E, making G_E an ample groupoid.

Define

$$A_R(G) = \{f : G \to R : f \text{ is continuous with compact support}\}.$$

We give $A_R(G)$ algebraic structure: define

- addition pointwise, then $A_R(G)$ becomes an R-module
- multiplication by

$$f * g(\gamma) = \sum_{r(\eta) = r(\gamma)} f(\eta) g(\eta^{-1}\gamma)$$

  (called a **convolution product**)

Then $A_R(G)$ is an R-algebra called the **Steinberg algebra** associated with G.

Since a Steinberg algebra is spanned by characteristic functions on compact open bisections, we have that

$$A_R(G_E) = \mathrm{span}_R\{1_{Z(\mu,F,\nu)} : \mu, \nu \in E^*, r(\mu) = r(\nu) \text{ and } F \subseteq r(\mu)E^1 \text{ is finite}\}.$$

Exercise 1. In $A_R(G_E)$, consider the functions as follows:

$$\begin{array}{ll} \text{for } v \in E^0 : & p_v = 1_{Z(v,v)} \\ \text{for } e \in E^1 : & s_e = 1_{Z(e,r(e))} \\ \text{for } e \in E^1 : & s_{e^*} = 1_{Z(r(e),e)} \end{array}$$

Prove that the following hold in $A_R(G_E)$

- (V) for $v, w \in E^0$, $p_v * p_v = p_v$ and $p_v * p_w = 0$ if $v \neq w$,
- (E1) for $e \in E^1$, $p_{s(e)} * s_e = s_e$ and $s_e * p_{r(e)} = s_e$,
- (E2) for $e \in E^1$, $p_{r(e)} * s_{e^*} = s_{e^*}$ and $s_{e^*} * p_{r(e)} = s_e^*$ (it is analogous to (E1)),
- (CK1) for $e, f \in E^1$, $s_{e^*} * s_e = p_{r(e)}$ and $s_{e^*} s_f = 0$ if $e \neq f$,
- (CK2) for $v \in E^0$ such that $0 < |vE^1| < \infty$ (that is, v is not a sink nor an infinite emitter),
$$p_v = \sum_{e \in s^{-1}(v)} s_e * s_{e^*}.$$

2. A dynamical point of view for the LPA relations

Exercise 2. For a given set X, let $\mathcal{I}(X) = \{f : A \to B \mid A, B \subseteq X \text{ and } f \text{ is a bijection}\}$ (the empty function is a bijection!).

(a) For $f : A \to B, g : C \to D \in \mathcal{I}(X)$, let $g \circ f : f^{-1}(B \cap C) \to g(B \cap C)$ be given by $(g \circ f)(x) = g(f(x))$, where $x \in X$. Prove that $g \circ f \in \mathcal{I}(X)$.

(b) For $f : A \to B, g : C \to D \in \mathcal{I}(X)$ such that $A \cap C = \emptyset = B \cap D$, show that there is natural way to define $f \cup g \in \mathcal{I}(X)$ (one can actually define $f \cup g$ whenever $f \circ g^{-1}$ and $f^{-1} \circ g$ are identity maps or the empty function).

Back to graphs, let $\partial E^{\geq 1} = \{\mu \in \partial E : |\mu| \geq 1\}$. We define a map $\sigma : \partial E^{\geq 1} = \partial E$ as follows

$$\begin{cases} \sigma(e) = r(e), & \text{if } e \in E^1 \cap \partial E \\ \sigma(e\mu) = \mu, & \text{if } e \in E^1 \text{ and } \mu \in \partial E^{\geq 1}. \end{cases}$$

The map σ is called the **shift map**. This gives a partially defined dynamics on ∂E.

Exercise 3. Consider the following functions:

- $P_v = Id_{Z(v)} : Z(v) \to Z(v)$ for $v \in E^0$,
- $S_e : Z(e) \to Z(r(e))$ given by $S_e(\nu) = \sigma(\nu)$ if $\nu \in Z(e)$, where $e \in E^1$.

(a) Prove that $P_v \in \mathcal{I}(\partial E)$ and $S_e \in \mathcal{I}(\partial E)$ for every $v \in E^0$ and $e \in E^1$. Describe S_e^{-1}

(b) Prove that the following holds in $\mathcal{I}(\partial E)$
- (V) for $v, w \in E^0$, $P_v \circ P_v = P_v$ and $P_v \circ P_w = \emptyset$ if $v \neq w$,
- (E1) for $e \in E^1$, $P_{s(e)} \circ S_e = S_e$ and $S_e \circ P_{r(e)} = S_e$,
- (E2) for $e \in E^1$, $P_{r(e)} \circ S_e^{-1} = S_e^{-1}$ and $S_e^{-1} \circ P_{r(e)} = S_e^{-1}$,
- (CK1) for $e, f \in E^1$, $S_e^{-1} \circ S_e = P_{r(e)}$ and $S_e^{-1} \circ S_f = \emptyset$ if $e \neq f$,
- (CK2) for $v \in E^0$ such that $0 < |vE^1| < \infty$ (that is, v is not a sink nor an infinite emitter),
$$P_v = \bigcup_{e \in s^{-1}(v)} S_e \circ S_e^{-1}.$$

(c) Prove that each S_e is a homeomorphism, so that σ is a local homeomorphism.