On groupoid algebras with applications to Leavitt labelled path algebras - part 4

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Free groups

Given a set $\mathcal{A}$, let us describe the construction of the free group $\mathbb{F}_{\mathcal{A}}$ generated by $\mathcal{A}$.

- If $\mathcal{A} = \emptyset$, we let $\mathbb{F}_{\emptyset}$ be the trivial group.
- For $\mathcal{A} \neq \emptyset$, we consider a set of “inverses” for $\mathcal{A}$. It is just a set denoted by $\mathcal{A}^{-1} = \{a^{-1} \mid a \in \mathcal{A}\}$ such that $\mathcal{A} \cap \mathcal{A}^{-1} = \emptyset$.
- $(\mathcal{A} \cup \mathcal{A}^{-1})^*$ denotes the set of all words with letters in $\mathcal{A} \cup \mathcal{A}^{-1}$, including the empty word ω.
- A word in $(\mathcal{A} \cup \mathcal{A}^{-1})^*$ is said to be reduced if no letter in $\mathcal{A}$ is adjacent to its inverse.
- We let $\mathbb{F}_{\mathcal{A}}$ be the set of all reduced words, which includes the empty word.

Example

Let $\mathcal{A} = \{a, b, c\}$, then $aab^{-1}b^{-1}a^{-1}c \in \mathbb{F}_{\mathcal{A}}$, but $aab^{-1}ba^{-1}c \notin \mathbb{F}_{\mathcal{A}}$.

We now describe the product in $\mathbb{F}_{\mathcal{A}}$:

- The empty word ω is the identity of $\mathbb{F}_{\mathcal{A}}$.
- The product of two reduced words is the concatenation follows by simplifications if needed.

Example

Let $\mathcal{A} = \{a, b, c\}$, $x = aab^{-1}$ and $y = ba^{-1}c$, then

$$xy = aab^{-1}ba^{-1}c$$

$$xy = aaa^{-1}c$$

$$xy = ac.$$

Remark

- Every element of $(\mathcal{A} \cup \mathcal{A}^{-1})^*$ can be seen as an element of $\mathbb{F}_{\mathcal{A}}$ by interpreting concatenation as the multiplication of $\mathbb{F}_{\mathcal{A}}$.
- There is length function $|\cdot| : \mathbb{F}_{\mathcal{A}} \to \mathbb{N}$ that returns the number of letters in a reduced word. The empyt word has length 0.

Group actions

Let X be a set and G a group. A function

$$\begin{array}{rccc} \cdot : & G \times X & \longrightarrow & X \\ & (g, x) & \longmapsto & g \cdot x \end{array}$$

is said to be a group action if

- $e \cdot x = x$ for all $x \in X$, where e is the identity of the group,
- $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$ and $g, h \in G$.

Alternatively, a group action can be described as a family of functions $\{\eta_g : X \to X\}_{g \in G}$ such that

- $\eta_e = Id_X$;
- $\eta_g \circ \eta_h = \eta_{gh}$ for all $g, h \in G$.

We identify the two approaches via the equation $\eta_g(x) = g \cdot x$ for all $x \in X$ and $g \in G$.

Partial actions

Definition

A partial action of a group G on a set X is a pair $\Phi = (\{U_t\}_{t\in G}, \{\phi_t\}_{t\in G})$ consisting of a collection $\{U_t\}_{t\in G}$ of subsets of X and a collection $\{\phi_t\}_{t\in G}$ of bijections, $\phi_t : U_{t^{-1}} \to U_t$, such that

(i) $U_e = X$ and ϕ_e is the identity on X,

(ii) $\phi_s(U_{s^{-1}} \cap U_t) = U_s \cap U_{st}$,

(iii) $\phi_s(\phi_t(x)) = \phi_{st}(x)$ for every $x \in U_{t^{-1}} \cap U_{(st)^{-1}}$.

If the partial action is given by the free group $\mathbb{F}$ on a set of generators, then the partial action is semi-saturated if

$$\phi_s \circ \phi_t = \phi_{st}$$

for every $s, t \in \mathbb{F}$ such that $|st| = |s| + |t|$, and orthogonal if $U_a \cap U_b = \emptyset$ for a, b in the set of generator with $a \neq b$.

Examples of partial actions

Example (Restriction of actions)

Suppose that $\eta = \{\eta_g : Y \to Y\}_{g\in G}$ is an action of a group G on a set Y. Given $X \subseteq Y$, we will restrict η to X. Unless X is G-invariant (ie, $\eta_g(X) = X$ for all $g \in G$), we do not obtain an action. Nevertheless, we always obtain a partial action as follows: for each $t \in G$, we define $U_t = X \cap \eta_t(X)$ and $\phi_t : U_{t^{-1}} \to U_t$ by $\phi_t(x) = \eta_t(x)$. Then $(\{U_t\}_{t\in G}, \{\phi_t\}_{t\in G})$ is a partial action.

Example (Graphs)

Let $\mathcal{E}$ be a graph and $\partial\mathcal{E}$ its boundary path space. Let also $\mathbb{F}$ be the free group generated by $\mathcal{E}^1$. We identify finite paths as elements of $\mathbb{F}$. We will define a partial action $(\{U_t\}_{t\in\mathbb{F}}, \{\phi_t\}_{t\in\mathbb{F}})$ of $\mathbb{F}$ on $\partial\mathcal{E}$, which is orthogonal and semi-saturated. The main idea is that $e \in E^1$ acts by adding e at the beginning of path and e^{-1} acts by removing e from the beginning of a path.

Example (Graphs - continued)

We divide the definition of U_t in a few cases:

- $U_\omega = \partial\mathcal{E}$,
- $U_\alpha = \{\alpha\mu \in \partial\mathcal{E}\}$ for $\alpha \in \mathcal{E}^{\geq 1}$,
- $U_{\beta^{-1}} = \{\mu \in \partial\mathcal{E} \mid s(\mu) = r(\beta)\}$ for $\beta \in \mathcal{E}^{\geq 1}$,
- $U_{\alpha\beta^{-1}} = \{\alpha\mu \in \partial\mathcal{E}\}$ for $\alpha, \beta \in \mathcal{E}^{\geq 1}$ such that $r(\alpha) = r(\beta)$ and $\alpha\beta^{-1}$ is in reduced form in $\mathbb{F}$,
- $U_t = \emptyset$ for all other cases.

Similarly we divide the definition of ϕ_t in a few cases

- $\phi_\omega = Id_{\partial\mathcal{E}}$,
- $\phi_\alpha(\mu) = \alpha\mu$ for $\alpha \in \mathcal{E}^{\geq 1}$,
- $\phi_{\beta^{-1}}(\beta\mu) = \mu$ for $\beta \in \mathcal{E}^{\geq 1}$,
- $\phi_{\alpha\beta^{-1}}(\beta\mu) = \alpha\mu$ for $\alpha, \beta \in \mathcal{E}^{\geq 1}$ such that $r(\alpha) = r(\beta)$ and $\alpha\beta^{-1}$ is in reduced form in $\mathbb{F}$,
- ϕ_t is the empty function for all other cases.

The transformation groupoid

Let $\Phi = (\{U_t\}_{t\in G}, \{\phi_t\}_{t\in G})$ be a partial action of G on a set X. Define the transformation groupoid as the set

$$G \ltimes_\Phi X = \{(x,t,y) \in X \times G \times X \mid y \in U_{t^{-1}} \text{ and } x = \phi_t(y)\}$$

with operations

$$(x,t,y)(y,s,z) = (x,ts,z)$$

and

$$(x,t,y)^{-1} = (y,t^{-1},y),$$

where $(x,t,y),(y,s,z) \in G \ltimes_\Phi X$.

Remark

Note that the information of a triple $(x,t,y) \in G \ltimes_\Phi X$ is completely encoded in the pair (t,y) or (x,t).

Topological partial actions

Suppose now that X is a topological space and for simplicity suppose that G is a discrete group. A topological partial action of G on X is a partial action $\Phi = (\{U_t\}_{t\in G}, \{\phi_t\}_{t\in G})$ such that for all $t \in G$, U_t is open and ϕ_t is a homeomorphism.

On $G \ltimes_\Phi X$ we put the topology induced by the product topology $X \times G \times X$. If X is a Stone space, then $G \ltimes_\Phi X$ is a Hausdorff ample groupoid.

Example

The partial action of $\mathbb{F}$ on $\partial\mathcal{E}$ defined above is a topological partial action and $\mathbb{F} \ltimes_\Phi \partial\mathcal{E}$ is a Hausdoff ample groupoid. In fact it isomorphic to graph groupoid $\mathcal{G}_\mathcal{E}$ seen in part 1 via $(\alpha\mu, \alpha\beta^{-1}, \beta\mu) \mapsto (\alpha\mu, |\alpha| - |\beta|, \beta\mu)$.

Algebraic partial actions

Let R be a commutative unital ring and $\mathcal{A}$ an R-algebra. We say that a partial action $\tau = (\{D_t\}_{t\in G}, \{\tau_t\}_{t\in G})$ of a group G on $\mathcal{A}$ is an algebraic partial action if for every $t \in G$, D_t is a (two-sided) ideal of $\mathcal{A}$ and τ_t is an isomorphism of R-algebras.

We will define an R-algebra from an algebraic partial action. This algebra is sometimes called a partial skew group ring and other times an (algebraic) partial crossed product.

As an R-module, we define

$$\mathcal{A} \rtimes_\tau G = \bigoplus_{t\in G} D_t.$$

An element of $\mathcal{A} \rtimes_\tau G$ will be written as a finite sum $\sum_{t\in G} a_t\delta_t$, where $a_t \in D_t$ and δ_t can be thought as a symbol to indicate the coordinate t.

In order to define the product in $\mathcal{A} \rtimes_\tau G$, it is enough to understand what happens for elements of the form $a_s\delta_s$ and $b_t\delta_t$ and then extend it in a bilinear way.

If we were to copy partial group rings, we would define the product as

$$a_s\delta_s \cdot b_t\delta_t = a_s\tau_s(b_t)\delta_{st},$$

however, we do not know whether $b_t \in D_{s^{-1}}$. To fix this, we observe that we can apply $\tau_{s^{-1}}$ on a_s and that $\tau_{s^{-1}}(a_s) \in D_{s^{-1}}$. Since $D_{s^{-1}}$ is an ideal, we have that $\tau_{s^{-1}}(a_s)b_t \in D_{s^{-1}}$, on which we can apply τ_s. We then define the product as

$$a_s\delta_s \cdot b_t\delta_t = \tau_s(\tau_{s^{-1}}(a_s)b_t)\delta_{st}.$$

Remark

This product is not always associative.

Theorem

Let $\tau = (\{D_t\}_{t\in G}, \{\tau_t\}_{t\in G})$ be an algebraic partial action such that D_t is idempotent for all $t \in G$ (that is, $D_tD_t = D_t$). Then the product on $\mathcal{A} \rtimes_\tau G$ defined above is associative.

Definition

Let S be a ring and F a family of idempotents of S. We say that F is a family of local units for S if for every $n \in \mathbb{N}$ and $s_1, \ldots, s_n$, there exists $f \in F$ such that $fs_i = s_if = s_i$ for all $i \in \{1, \ldots, n\}$.

Remark

If S has a family of local units then S is idempotent.

Dual partial action

Let G be a discrete group and X be a Hausdorff space with a basis of compact-set (ie, a Stone space). Given a topological partial action $\Phi = (\{U_t\}_{t\in G}, \{\phi_t\}_{t\in G})$ of G on X, suppose that U_t is clopen for all $t \in G$. From this, we can build an algebraic partial action of G on $\mathrm{Lc}(X, R)$, which we call a dual partial action.

For each $t \in G$, we set $D_t = \mathrm{Lc}(U_t, R)$ seen as an ideal of $\mathrm{Lc}(X, R)$ by a extending a function $f : U_t \to R$ to $f : X \to R$ by defining it to be 0 outside U_t. Because U_t is clopen this extension is indeed in $\mathrm{Lc}(X, R)$. We observe that $F = \{1_V \mid V \subseteq U_t \text{ is compact-open}\}$ is a family of local units for D_t.

We also define $\hat{\phi}_t : D_{t^{-1}} \to D_t$ by $\hat{\phi}_t(f) = f \circ \phi_{t^{-1}}$.

Proposition

$\hat{\Phi} = (\{D_t\}_{t\in G}, \{\hat{\phi}_t\}_{t\in G})$ defined as above is an algebraic partial action such that D_t is idempotent for all t.

Theorem

With the above construction $A_R(G \ltimes_\Phi X) \cong \mathrm{Lc}(X, R) \rtimes_{\hat{\Phi}} G$.

Idea of the proof.

Given $f \in A_R(G \ltimes_\Phi X)$, for each $t \in G$, we define $f_t : U_t \to R$ by $f_t(x) = f(x, t, \phi_{t^{-1}}(x))$. The map given by

$$f \mapsto \sum_{t \in G} f_t \delta_t$$

is then a well-defined homomorphism of R-algebras.

For the inverse, we send a sum $\sum_{t \in G} f_t \delta_t$ to the function $f : G \ltimes_\Phi X \to R$ given by $f(x, t, y) = f_t(x)$. □

Partial actions from labelled spaces

Given a labelled space $(\mathcal{E}, \mathcal{L}, \mathcal{B})$, that we have a topological correspondence (E^1, s, r) from F^0 to E^0, where $E^0 = X_\omega$, $F^0 = X_\omega \cup \{\emptyset\}$ its one-point extension and $E^1 = \bigsqcup_{a\in\mathcal{A}} X_a$ with the disjoint topology. We denote an element of E^1 by $e^a_{\mathcal{F}}$ for $a \in \mathcal{A}$ and $\mathcal{F} \in X_a$. And we have a boundary path space ∂E.

Analogous to the graph case we can define

- $U_\omega = \partial E$,
- $U_\alpha = \{e^{\alpha_1}_{\mathcal{F}_1} \cdots e^{\alpha_n}_{\mathcal{F}_n}\mu \in \partial E\}$ for $\alpha \in \mathcal{L}^n$ with $n \geq 1$,
- $U_{\beta^{-1}} = \{\mu \in \partial E \mid r(\beta) \in s(\mu)\}$ for $\beta \in \mathcal{L}^{\geq 1}$,
- $U_{\alpha\beta^{-1}} = \{e^{\alpha_1}_{\mathcal{F}_1} \cdots e^{\alpha_n}_{\mathcal{F}_n}\mu \in \partial E \mid r(\beta) \in s(\mu)\}$ for $\alpha, \beta \in \mathcal{L}^{\geq 1}$ such that $|\alpha| = n$, $r(\alpha) \cap r(\beta) \neq \emptyset$ and $\alpha\beta^{-1}$ is in reduced form in $\mathbb{F}$,
- $U_t = \emptyset$ for all other cases.

Remark

For all $t \in \mathbb{F} \setminus \{\omega\}$, we have that U_t is compact-open.

For the maps, we can prove that $\phi_{\alpha^{-1}} : U_\alpha \to U_{\alpha^{-1}}$ given by $\phi_{\alpha^{-1}}(e^{\alpha_1}_{\mathcal{F}_1} \cdots e^{\alpha_n}_{\mathcal{F}_n}\mu) = \mu$ is a well-defined homeomorphism.

Because of the filters $\mathcal{F}_1, \ldots, \mathcal{F}_n$ the description of the inverse is much more technical. Nevertheless, we can define $\phi_\alpha = (\phi_{\alpha^{-1}})^{-1}$ and $\phi_{\alpha\beta^{-1}} = \phi_\alpha \circ \phi_{\beta^{-1}}$, whenever it makes sense.

Theorem

The construction above results in an orthogonal semi-saturated partial action Φ of $\mathbb{F}$ on ∂E such that $\Gamma(E) \cong \mathbb{F} \ltimes_\Phi \partial E$.

Corollary

$L_R(\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong \mathrm{Lc}(\partial E, R) \rtimes_{\hat{\Phi}} \mathbb{F}$.

Theorem

Let X be a Stone space and let $\rho = (\{V_t\}_{t\in\mathbb{F}}, \{\rho_t\}_{t\in\mathbb{F}})$ be a semi-saturated, orthogonal topological partial action of a free group $\mathbb{F}$ on X such that V_t is compact-open for all $t \in \mathbb{F} \setminus \{\omega\}$. Then there exists a labelled space $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ and a homeomorphism $f : X \to \partial E$, where ∂E, such that f is equivariant with respect to the actions ρ and Φ given by the above theorem. In particular $\mathbb{F} \ltimes_\rho X$ and $\mathbb{F} \ltimes_\varphi \partial E$ are isomorphic as topological groupoids.

Idea of the proof.

- $\mathcal{A}$ is the set of generators of $\mathbb{F}$,
- $\mathcal{E}^0 = X$,
- $\mathcal{E}^1 = \{(a, x) \in \mathcal{A} \times X \mid x \in V_a\}$,
- $s(a, x) = x$, $r(a, x) = \rho_{a^{-1}}(x)$,
- $\mathcal{L}(a, x) = a$,
- $\mathcal{B}$ is the set of compact-open subsets of X.

□

Corollary

Let R be a commutative unital ring and $\mathcal{A}$ be a torsion-free commutative R-algebra generated by its idempotents elements. Let also $\tau = (\{D_t\}_{t\in\mathbb{F}}, \{\tau_t\}_{t\in\mathbb{F}})$ be a semi-saturated, orthogonal algebraic partial action of a free group $\mathbb{F}$ on $\mathcal{A}$ such that D_t is unital for every $t \in \mathbb{F} \setminus \{\omega\}$. Then, there exists a labelled space $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ such that $\mathcal{A} \rtimes_\tau \mathbb{F} \cong L_R(\mathcal{E}, \mathcal{L}, \mathcal{B})$.

Idea of the proof.

We have that $\mathcal{A} \cong \mathrm{Lc}(X, R)$ for some Stone space X. For $t \in \mathbb{F} \setminus \{\omega\}$, because D_t is unital, we have $D_t \cong \mathrm{Lc}(U_t, R)$ for some compact-open U_t subset of X.

We can then define partial action $\rho = (\{U_t\}_{t\in\mathbb{F}}, \{\rho_t\}_{t\in\mathbb{F}})$ such that its dual $\hat{\rho}$ is essentially the same as τ. Hence, for the labelled space $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ of the previous theorem, we get

$$\mathcal{A} \rtimes_\tau \mathbb{F} \cong \mathrm{Lc}(X, R) \rtimes_{\hat{\rho}} \mathbb{F} \cong L_R(\mathcal{E}, \mathcal{L}, \mathcal{B}).$$

$\square$

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Thank you!