# On groupoid algebras with applications to Leavitt labelled path algebras - part 3

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# Cuntz-Krieger algebras and generalisations

- Cuntz algebras (1977) - Simple infinite C*-algebras.
- Cuntz-Krieger algebras (1980) - C*-algebras for topological Markov chains.
- Graph C*-algebras (Kumjian, Pask, Raeburn, Renault - 1997).
- C*-Algebras for two-sided subshifts (Matsumoto 1997, Carlsen-Matsumoto 2004)
- Exel-Laca algebras for infinite matrices of 0-1 (1999).
- Ultragraph C*-algebras (Tomforde 2003).
- C*-algebras of labelled graphs (Bates, Pask 2007).
- C*-algebras for one-sided subshifts (Carlsen 2008).
- C*-algebras of Boolean dynamical systems (Carlsen, Ortega, Pardo - 2017).
- C*-algebras of generalised Boolean dynamical systems (Carlsen, Kang - 2020).

# Leavitt path algebras

- Leavitt rings/algebras (Late 1950s, early 1960s) - rings without the IBN property.
- Purely algebraic analogue of Cuntz-Krieger algebras (Ara, Gonzáles-Barros, Goodearl, Pardo - 2004).
- Leavitt path algebras for graphs (Ara, Pino - 2005).
- Algebras for Boolean dynamical systems (Clark, Exel, Pardo - 2018).
- Leavitt path algebras for ultragraphs (Imanfar, Pourabbas, Larki - 2020).
- Leavitt path algebras for labelled graphs (Boava, de C., Gonçalves, van Wyk - 2021*) - includes both Leavitt path algebras for graphs and commutative algebras generated by idempotents.

# Boolean algebras

A (concrete) Boolean algebra is a non-empty family $\mathcal{B}$ of subsets of a given set $X$ such that

- $A \cup B \in \mathcal{B}$,
- $A \cap B \in \mathcal{B}$,
- $A \setminus B \in \mathcal{B}$.

for all $A, B \in \mathcal{B}$.

## Remark

- $\emptyset \in \mathcal{B}$ because $A \setminus A = \emptyset$ for a given $A \in \mathcal{B}$.
- We do not assume here that $X \in \mathcal{B}$ as it is usually asked in the definition of a Boolean algebra. The definition above is then sometimes called a generalized Boolean algebra.

## Example

Let $X$ be any set and $\mathcal{B}$ be the family of all finite subsets of $X$. Then $\mathcal{B}$ is a Boolean algebra.

### Example

Let $X$ be a Hausdorff space. The family $\mathcal{B}$ of compact-open subsets of $X$ is a Boolean algebra. This happens because in a Hausdorff spaces, compact subsets are closed.

### Remark

There are other ways of defining Boolean algebras:

- Algebraically: A Boolean algebra is a set $\mathcal{B}$ with binary operations $\cup, \cap, \setminus$ satisfying a certain list of axioms.
- Using order theory: A Boolean algebra is a relatively complemented distributive lattice with least element.

# Stone duality

**Theorem**

*Every Boolean algebra is isomorphic to the Boolean algebra of compact-open subsets of a Hausdorff space with a basis of compact-open sets.*

**Idea of the proof.**

Let $\mathcal{B}$ be a Boolean algebra. A filter in $\mathcal{B}$ is a set $\mathcal{F} \subseteq \mathcal{B}$ such that $A \cap B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$ and whenever $B \in \mathcal{B}$ is such that $A \subseteq B$ for some $A \in \mathcal{F}$, we have that $B \in \mathcal{F}$. An ultrafilter in $\mathcal{B}$ is a proper maximal filter.

The Stone dual of $\mathcal{B}$ is the set $X$ of all ultrafilters with a basis of compact-open sets given by sets of the form $U_A = \{\mathcal{F} \in X \mid A \in \mathcal{F}\}$ for $A \in \mathcal{B}$. In fact, all compact-open sets of $X$ are of the form $U_A$ for some $A$ and the map $A \mapsto U_A$ is a Boolean algebra isomorphism. $\square$

## Theorem

*If $X$ is Hausdorff space with a basis of compact-open set and $\mathcal{B}$ is the Boolean algebra of compact-open subsets of $X$ then $X$ is homeomorphic to the Stone dual of $\mathcal{B}$.*

## Idea of the proof.

Given $x \in X$, the set $\mathcal{F}_x = \{A \in \mathcal{B} \mid x \in A\}$ is an ultrafilter.

Because $\mathcal{B}$ is a basis for $X$, every ultrafilter of $\mathcal{B}$ is of the form $\mathcal{F}_x$ for some $x$.

The homeomorphism is given by $x \mapsto \mathcal{F}_x$. $\square$

## Definition

A Hausdorff space with a basis of compact-open sets will be called a **Stone space**.

# Commutative algebras generated by idempotents

Let $R$ be a commutative unital ring and $\mathcal{A}$ a commutative $R$-algebra. On the set $E(\mathcal{A})$ of idempotents of $\mathcal{A}$ we can define a structure of Boolean algebra by

- $e \cup f = e + f - ef$,
- $e \cap f = ef$,
- $e \setminus f = e - ef$,

for $e, f \in E(\mathcal{A})$.

**Theorem**

*Suppose that $\mathcal{A}$ is generated by $E(\mathcal{A})$ and that for all $r \in R$ and $e \in E(\mathcal{A})$ we have that $re = 0$ implies $r = 0$ or $e = 0$. Let $X$ be the Stone dual of $E(\mathcal{A})$, then $\mathcal{A} \cong \mathrm{Lc}(X, R)$, where $\mathrm{Lc}(X, R)$ is the set of locally constant functions from $X$ to $R$ with compact support.*

# A universal property

## Theorem

*Let $\mathcal{B}$ be a Boolean algebra with Stone dual $X$ and $R$ a commutative unital ring. Then, $\mathrm{Lc}(X, R)$ is isomorphic to the universal $R$-algebra generated by a family $\{p_A \mid A \in \mathcal{B}\}$ such that*

- $p_\emptyset = 0$,
- $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ *for all* $A, B \in \mathcal{B}$,
- $p_{A\cap B} = p_A p_B$ *for all* $A, B \in \mathcal{B}$.

## Remark

That $p_{A\setminus B} = p_A - p_{A\cap B}$ for all $A, B \in \mathcal{B}$ follows as a consequence of the other relations.

## Example

Suppose that $X$ has the discrete topology. For $x \in X$, let $p_x = p_{\{x\}}$. Then $p_x p_y = 0$ if $x \neq y$. A compact-open set $A$ of $X$ is just a finite set. If $A = \{x_1, \ldots, x_n\}$, then $p_A = p_{x_1} + \cdots + p_{x_n}$.

# Labelled graphs

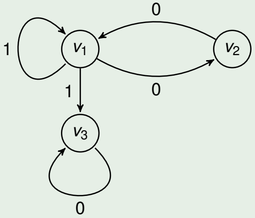
- By a (directed) graph we mean a quadruple $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ where $\mathcal{E}^0, \mathcal{E}^1$ are sets, $s : \mathcal{E}^1 \to \mathcal{E}^0$ and $r : \mathcal{E}^1 \to \mathcal{E}^0$ are maps.
- Given a set $\mathcal{A}$, which is thought as a set of letters, an (edge-)labelling on a graph $\mathcal{E}$ is an onto map $\mathcal{L} : \mathcal{E}^1 \to \mathcal{A}$.
- We call the pair $(\mathcal{E}, \mathcal{L})$ a labelled graph.
- A path $\lambda$ on $\mathcal{E}$ is a sequence (finite or infinite) of edges $\lambda = \lambda_1 \dots \lambda_n (\dots)$ such that $r(\lambda_i) = s(\lambda_{i+1})\ \forall i$.
- We can extend the map $\mathcal{L}$ to any path $\lambda$ by $\mathcal{L}(\lambda) = \mathcal{L}(\lambda_1) \dots \mathcal{L}(\lambda_n)(\dots)$.
- An element $\alpha = \mathcal{L}(\lambda)$ is called a labelled path. We also include the empty word $\omega$ as a labelled path.
- For $A \subseteq \mathcal{E}^0$, we define $\mathcal{L}(A\mathcal{E}^1) = \{\mathcal{L}(e) \mid e \in \mathcal{E}^1,\ s(e) \in A\}$.

For $\alpha \in \mathcal{L}^*$ and $A \in \mathcal{P}(\mathcal{E}^0)$, the relative range of $\alpha$ with respect to $A$ is

$$r(A, \alpha) = \{r(\lambda) \mid \lambda \in \mathcal{E}^*,\ \mathcal{L}(\lambda) = \alpha,\ s(\lambda) \in A\}$$

if $\alpha \in \mathcal{L}^{\geq 1}$ and $r(A, \omega) = A$ if $\alpha = \omega$. We define $r(\alpha) := r(\mathcal{E}^0, \alpha)$.

## Example




- The labelled paths are all finite and infinite sequences of 0's and 1's such that between two 1's there must be an even number of 0's (even shift).
- $r(\{v_1, v_2\}, 1) = \{v_1, v_3\}$, $\quad r(\{v_3\}, 1) = \emptyset$, $\quad r(1) = \{v_1, v_2\}$, $\mathcal{L}(\{v_2, v_3\}\mathcal{E}^1) = \{0\}$.

# Labelled spaces

## Definition

A (normal weakly left-resolving) labelled space is a triple $(\mathcal{E}, \mathcal{L}, \mathcal{B})$, where $(\mathcal{E}, \mathcal{L})$ is a labelled graph and $\mathcal{B} \subseteq \mathcal{P}(\mathcal{E}^0)$ is a Boolean algebra such that

- $r(\alpha) \in \mathcal{B}$ and $r(A, \alpha) \in \mathcal{B}$ for all $\alpha \in \mathcal{L}^{\geq 1}$ and $A \in \mathcal{B}$,
- $r(A \cap B, \alpha) = r(A, \alpha) \cap r(B, \alpha)$ for all $\alpha \in \mathcal{L}^{\geq 1}$ and $A, B \in \mathcal{B}$.

For each $A \in \mathcal{B}$, we let $\Delta_A = \{a \in \mathcal{A} \mid r(A, a) \neq \emptyset\}$. We say that $A \in \mathcal{B}$ is regular if for all $B \in \mathcal{B} \setminus \{\emptyset\}$ such that $B \subseteq A$, we have that $0 < |\Delta_B| < \infty$. The set of regular sets is denoted by $\mathcal{B}_{reg}$.

# Leavitt path algebras for labelled graphs

**Definition**

Let $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ be a labelled space and $R$ a unital commutative ring. The Leavitt labelled path algebra associated with $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ with coefficients in $R$, denoted by $L_R(\mathcal{E}, \mathcal{L}, \mathcal{B})$, is the universal $R$-algebra with generators $\{p_A \mid A \in \mathcal{B}\}$ and $\{s_a, s_a^* \mid a \in \mathcal{A}\}$ subject to the relations

(i) $p_{A\cap B} = p_A p_B$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ and $p_\emptyset = 0$, for every $A, B \in \mathcal{B}$;

(ii) $p_A s_a = s_a p_{r(A,a)}$ and $s_a^* p_A = p_{r(A,a)} s_a^*$, for every $A \in \mathcal{B}$ and $a \in \mathcal{A}$;

(iii) $s_a^* s_a = p_{r(a)}$ and $s_b^* s_a = 0$ if $b \neq a$, for every $a, b \in \mathcal{A}$;

(iv) $s_a s_a^* s_a = s_a$ and $s_a^* s_a s_a^* = s_a^*$ for every $a \in \mathcal{A}$;

(v) For every $A \in \mathcal{B}_{reg}$,

$$p_A = \sum_{a\in\mathcal{L}(A\mathcal{E}^1)} s_a p_{r(A,a)} s_a^*.$$

# Graded uniqueness theorem

The Leavitt labelled path algebra $L_R(\mathcal{E},\mathcal{L},\mathcal{B})$ is $\mathbb{Z}$-graded, with grading given by

$$L_R(\mathcal{E},\mathcal{L},\mathcal{B})_n = \mathrm{span}_R\{s_\alpha p_A s_\beta^* \mid \alpha,\beta\in\mathcal{L}^*,\ A\in\mathcal{B}_\alpha\cap\mathcal{B}_\beta,\ |\alpha|-|\beta|=n\}.$$

**Theorem (Graded uniqueness theorem)**

*If $A$ is a $\mathbb{Z}$-graded ring and $\phi : L_R(\mathcal{E},\mathcal{L},\mathcal{B}) \to A$ is a graded ring homomorphism with $\phi(rp_A)\neq 0$ for all non-empty $A\in\mathcal{B}$ and all non-zero $r\in R$, then $\phi$ is injective.*

# Example - graph algebras

## Proposition

*Let $\mathcal{E}$ be a graph and consider $\mathcal{A} = \mathcal{E}^1$, $\mathcal{L} = Id_{\mathcal{E}^1}$ and $\mathcal{B} = \{A \subseteq \mathcal{E}^0 \mid A \text{ is finite}\}$. Then $L_R(\mathcal{E}) \cong L_R(\mathcal{E}, \mathcal{L}, \mathcal{B})$.*

## Idea of the proof.

We define a map $\phi : L_R(\mathcal{E}, \mathcal{L}, \mathcal{B}) \to L_R(\mathcal{E})$ by

- $\phi(p_A) = \sum_{v \in A} v$ for all $A \in \mathcal{B}$,
- $\phi(s_e) = e$, $\phi(s_e^*) = e^*$ for all $e \in \mathcal{E}^1$.

To show that $\phi$ is well-defined, we have to prove that $\phi$ preserves the relations defining $L_R(\mathcal{E}, \mathcal{L}, \mathcal{B})$. Most of the computations are straightforward observing that $r(A, e) = r(e)$ if $s(e) \in A$ and $r(A, e) = \emptyset$ otherwise.

It is easy to see that the image of $\phi$ contains the generators of $L_R(\mathcal{E})$ so that $\phi$ is surjective. And injectivity follows from the graded uniqueness theorem. □

## Definition

Let $(\mathcal{E},\mathcal{L})$ be a labelled graph. We say that $(\mathcal{E},\mathcal{L})$ is left-resolving if for every $v\in\mathcal{E}^0$, we have that $\mathcal{L}|_{r^{-1}(v)}$ is injective. We say that $(\mathcal{E},\mathcal{L})$ is label-finite if $|\mathcal{L}^{-1}(a)|<\infty$ for every $a\in\mathcal{A}$.

## Proposition

*Let $(\mathcal{E},\mathcal{L})$ be a left-resolving, label-finite, labelled graph, and let $\mathcal{B}$ be the family of all finite subsets of $\mathcal{E}^0$. Then $L_R(\mathcal{E},\mathcal{L},\mathcal{B})\cong L_R(\mathcal{E})$.*

## Idea of the proof.

We define a map $\phi : L_R(\mathcal{E},\mathcal{L},\mathcal{B})\to L_R(\mathcal{E})$ by

- $\phi(p_A)=\sum_{v\in A} v$ for all $A\in\mathcal{B}$,
- $\phi(s_a)=\sum_{\mathcal{L}(e)=a} e$, $\phi(s_a^*)=\sum_{\mathcal{L}(e)=a} e^*$ for all $a\in\mathcal{A}$.

The property that $(\mathcal{E},\mathcal{L})$ is left-resolving is used to prove that $\phi$ is well-defined and surjective. $\square$

# Example - commutative algebras

**Proposition**

*Let $\mathcal{B}$ be a Boolean algebra and let $X$ be its Stone dual. Define $\mathcal{E}^0 = X$, $\mathcal{A} = \mathcal{E}^1 = \emptyset$ and $\mathcal{L}$ the empty function. Then $L_R(\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong \mathrm{Lc}(X, R)$.*

**Proposition**

*If $L_R(\mathcal{E})$ is commutative, then $L_R(\mathcal{E})$ is a direct sum of copies of $R$ and $R[x, x^{-1}]$.*

**Example**

Let $\overline{\mathbb{N}}$ be the one-point compactification of $\mathbb{N}$ and suppose that $R$ is an integral domain. Then $\mathrm{Lc}(\overline{\mathbb{N}}, R)$ is a Leavitt labelled path algebra that is not a Leavitt path algebra for any graph. Indeed $\mathrm{Lc}(\overline{\mathbb{N}}, R)$ cannot contain a copy of $R[x, x^{-1}]$ because it is generated by its idempotents. We also cannot have $\mathrm{Lc}(\overline{\mathbb{N}}, R) \cong \bigoplus_{i \in I} R$, because $\bigoplus_{i \in I} R \cong \mathrm{Lc}(I, R)$, where $I$ has the discrete topology.

# Groupoid model

Fix $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ a labelled space. For each $\alpha \in \mathcal{L}^*$, we have that $\mathcal{B}_\alpha = \{A \in \mathcal{B} \mid A \subseteq r(\alpha)\}$ is a Boolean algebra, which is unital whenever $\alpha \neq \omega$. We denote by $X_\alpha$ the corresponding Stone dual.

For each $a \in \mathcal{A}$, we define two maps $f_a : X_a \to X_\omega \cup \{\emptyset\}$ and $h_a : X_a \to X_\omega$ by

$$f_a(\mathcal{F}) = \{A \in \mathcal{B} \mid r(A, a) \in \mathcal{F}\}$$

and

$$h_a(\mathcal{F}) = \{A \in \mathcal{B} \mid \exists B \in \mathcal{F} \text{ s.t. } B \subseteq A\}$$

where $\mathcal{F} \in X_a$.

Let $E^0 = X_\omega$, $F^0 = X_\omega \cup \{\emptyset\}$ its one-point extension and $E^1 = \bigsqcup_{a\in\mathcal{A}} X_a$ with the disjoint topology. We denote an element of $E^1$ by $e^a_{\mathcal{F}}$ for $a \in \mathcal{A}$ and $\mathcal{F} \in X_a$. Define maps $s : E^1 \to F^0$ by $s(e^a_{\mathcal{F}}) = f_a(\mathcal{F})$ and $r : E^1 \to E^0$ by $r(e^a_{\mathcal{F}}) = h_a(\mathcal{F})$. Then $(E^1, s, r)$ is a topological correspondence from $F^0$ to $E^0$.

Because $E^0 \subseteq F^0$, we can define the set of finite paths $E^*$ and infinite paths $E^\infty$ in the usual way. Let $E^0_{reg} = \{v \in E^0 \mid \exists V \text{ cpt. ngbh. of } v \text{ s.t. } s^{-1}(V) \text{ is cpt. and } V = s(s^{-1}(V))\}$. The boundary path space is the set $\partial E = \{\mu \in E^* \mid r(\mu) \notin E^0_{reg}\} \cup E^\infty$.

We can then consider the boundary path groupoid

$$\Gamma(E) = \{(\mu\gamma, |\mu| - |\nu|, \nu\gamma) \in \partial E \times \mathbb{Z} \times \partial E \mid \mu, \nu \in E^*, \gamma \in \partial E\},$$

which is an amenable étale ample Hausdorff groupoid with an appropriate topology.

## Theorem

*Let* $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ *be a labelled space. Then* $L_R(\mathcal{E}, \mathcal{L}, \mathcal{B}) \cong A_R(\Gamma(E))$.

## Idea of the proof.

For $A \in \mathcal{B}$, we define

$$U_A = \{(\mu, 0, \mu) \in \Gamma(E) \mid A \in s(\mu)\},$$

and for $a \in \mathcal{A}$, we define

$$V_a = \{(e^a_{\mathcal{F}}\mu, 1, \mu) \in \Gamma(E)\}.$$

The isomorphism $\phi : L_R(\mathcal{E}, \mathcal{L}, \mathcal{B}) \to A_R(\Gamma(E))$ is given by $\phi(p_A) = 1_{U_A}$, $\phi(s_a) = 1_{V_a}$ and $\phi(s_a^*) = 1_{V_a^{-1}}$. $\square$

# References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Thank you!