Part 2: Introduction to Steinberg Algebras

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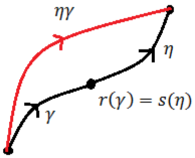
8 February 2022

Recap

A groupoid is set G with

- multiplication - only defined for composable pairs: (γ, η) such that $s(\gamma) = r(\eta)$.
- inverses

which satifies axioms similar to those of a group (when it is meaningful!)



• G is a topological groupoid if it has locally compact topology in which the multiplication and inverse maps are continuous.

Ample groupoids

Recall

- Open set $B \subset G$ is bisection if $r|_B$ and $s|_B$ are homeomorphism onto open subsets of $G^{(0)}$.
- G étale if and only if G has a basis of open bisections.
- G is ample if G has a basis of **compact open** bisections.

For the remainder of today's talk G denotes a Hausdorff ample groupoid.

Building blocks of Steinberg Algebras

- Let R be a unital commutative ring R with the discrete topology.

- We say $f : G \to R$ is locally constant if for every $\gamma \in G$ there is an open neigborhood U of γ such that $f|_U$ is constant.

- f is locally constant if and only if $f : G \to R$ is continuous. (($\Rightarrow$) holds for any topology on R, and ($\Leftarrow$) is because R has the discrete topology.)

- Let $f : G \to R$ be locally constant and define the support of f as

$$\mathrm{supp}(f) = \{\gamma \in G : f(\gamma) \neq 0\}$$

Note that $\mathrm{supp}(f)$ is clopen.

Steinberg Algebras over rings

Define

$$A_R(G) = \{f : G \to R : f \text{ is continuous with compact support}\}.$$

We give $A_R(G)$ algebraic structure: define

- addition pointwise, then $A_R(G)$ becomes an R-module
- multiplication by

$$f * g(\gamma) = \sum_{r(\eta)=r(\gamma)} f(\eta)g(\eta^{-1}\gamma)$$

(called a convolution product)

Then $A_R(G)$ is an R-algebra called the Steinberg algebra associated with G.

Steinberg Algebras over $\mathbb{C}$

If we take $R = \mathbb{C}$, then we can include an involution: let

$$A_{\mathbb{C}}(G) = \{f : G \to R : f \text{ has compact support}\}.$$

The Steinberg algebra $A_{\mathbb{C}}(G)$ is a *-algebra with

- pointwise addition and scalar multiplication,
- multiplication given by

$$f * g(\gamma) = \sum_{r(\eta)=r(\gamma)} f(\eta)g(\eta^{-1}\gamma),$$

- involution given by

$$f(\gamma)^* = \overline{f(\gamma^{-1})}.$$

• If R is a unital commutative ring with an involution, then $A_R(G)$ is also a *-algebra.

$A_R(G)$ in terms of characteristic functions

Let $B^{co}(G) = \{U \subset G : U \text{ is a compact open bisection in } G\}$.

Then $B^{co}(G)$ is an inverse-semigroup where

$$UV = \{\gamma\eta : \gamma \in U, \eta \in V\} \text{ and } U^{-1} = \{\gamma^{-1} : \gamma \in U\}$$

The set of idempotents of $B^{co}(G)$ is $B^{co}(G^{(0)})$.

For any $U \in B^{co}$ we define $1_U : G \to R$ by

$$1_U(\gamma) = \begin{cases} 1 & \text{if } \gamma \in U \\ 0 & \text{otherwise} \end{cases}$$

Clearly, 1_U is locally constant with compact support; thus $1_U \in A_R(G)$ for every $U \in B^{co}$.

Proposition

Let G be an ample groupoid and R a unital commutative ring. Then

$$A_R(G) = \mathrm{span}_R\{1_U : U \in B^{co}(G)\}.$$

Properties:
(i) for $U, V \in B^{co}(G)$, we have $1_U * 1_V = 1_{UV}$,
(multiplication in $A_R(G)$ corresponds to multiplication in B^{co})
To see this, consider

$$\begin{aligned} 1_U * 1_V(\gamma) &= \sum_{r(\eta)=r(\gamma)} 1_U(\eta) 1_V(\eta^{-1}\gamma) \\ &= \begin{cases} 1 & \text{if } \eta \in U, \text{ and } \eta^{-1}\gamma \in V \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Note $\eta \in U$ and $\eta^{-1}\gamma \in V$ imply $\gamma = \eta\eta^{-1}\gamma \in UV$.
On the other hand, if $\gamma \in UV$, then $\gamma = \eta\alpha$ where $\eta \in U$ and $\alpha \in V$. Then, $\eta^{-1}\gamma = \alpha$, and $s(\alpha) = r(\eta) = r(\gamma)$. That is $1_U * 1_V(\gamma) = 1$

Proposition

Let G be an ample groupoid and R a unital commutative ring. Then

$$A_R(G) = \mathrm{span}_R\{1_U : U \in B^{co}(G)\}.$$

Properties:

(i) for $U, V \in B^{co}(G)$, we have $1_U * 1_V = 1_{UV}$,

(multiplication in $A_R(G)$ corresponds to multiplication in B^{co})

(ii) if $R = \mathbb{C}$, then $1_U^* = 1_{U^{-1}}$,

(inversions in $A_R(G)$ corresponds to inversion in B^{co})

Proposition

Let G be an ample groupoid and R a unital commutative ring. Then

$$A_R(G) = \mathrm{span}_R\{1_U : U \in B^{co}(G)\}.$$

Properties:
(iii) For $U, V \in B^{co}(G^{(0)})$, we have $1_U * 1_V = 1_U 1_V = 1_{U\cap V}$ (convolution on the unit space is pointwise multiplication)
To see this, note that $U, V \subset B^{c0}(G^{(0)})$, then

$$UV = \{uv : u \in U, v \in V\}.$$

But u and v are units, implying that $r(u) = s(u) = r(v) = s(u)$. Therefore,

$$UV = U \cap V.$$

Then

$$1_U * 1_V = 1_{UV} = 1_{U\cap V} = 1_U 1_V.$$

Proposition

Let G be an ample groupoid and R a unital commutative ring. Then

$$A_R(G) = \mathrm{span}_R\{1_U : U \in B^{co}(G)\}.$$

Properties:

(iv) $A_R(G)$ is unital if and only if $G^{(0)}$ is compact, with $1_{G^{(0)}}$ being the unit.

(v) $A_R(G^{(0)}) \subset A_R(G)$

Examples

Matrix groupoid: Let $N \in \mathbb{N}$ and $R_N = \{1,2,\ldots,N\} \times \{1,2,\ldots,N\}$ the matrix groupoid (with discrete topology). Then

$$A_{\mathbb{C}}(R_N) = \mathrm{span}_{\mathbb{C}}\{1_{(i,j)} : 1 \leq i,j \leq N\}.$$

If $1_{(i,i)}, 1_{(k,l)} \in A_{\mathbb{C}}(R_N)$, then

$$1_{(i,j)} * 1_{(k,l)} = 1_{(i,j)(k,l)} = \delta_{j,k} 1_{(i,l)}.$$

Therefore, every $1_{(i,j)}$ is a matrix unit, e.g

$$1_{(2,3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

Hence $A_{\mathbb{C}}(R_N) \cong M_N(\mathbb{C})$.

Examples

Transformation groupoid: Soppose H discrete group (e.g $\mathbb{Z}$) that acts on a locally compact Hausdorff space X by homeomorphisms and that X has **basis of compact open sets $\mathcal{B}$.**

Step 1: An induced action on and algebra:

Suppse the action is given by $\phi : H \to Homeo(X), h \mapsto \phi_h$.

The skew group ring associted with (H, X) is a non-commutative ring that captures the dynamics. It is constructed as follows:

Let $L_c(X) = \{f : X \to R : f$ is locally constant$\}$. Then $L_c(X)$ is an R-algebra with pointwise operations.

Define $\hat{\phi} : H \to Aut(L_c(X))$ by

$$\hat{\phi}_h(f)(x) = f \circ \phi_h(x).$$

Then $\hat{\phi}$ is an action of H on $L_c(X)$.

Examples

Transformation groupoid:

Step 2: The skew group ring: let

$$L_c(X) \rtimes_{\hat{\phi}} H = \left\{ \sum_{h \in H} a_h \delta_h : a_h \in L_c(X) \text{ and at most fintely many } a_h \neq 0 \right\}$$

and define

- addition is defined pointwise
- mulitplication is defined by

$$(a_g \delta_g)(a_h \delta_h) = a_g \hat{\phi}_h(a_h) \delta gh$$

- $(a_h \delta_h)^* = a_h^* \delta_{h^{-1}}$ (if R has an involution)

Examples

3. Transformation groupoid:
Step 3: The Steinberg algebra of

$$G = \{(x, h, y) \in X \times H \times X : x = \phi_h(y)\}$$

is given by

$$A_R(G) = \mathrm{span}_R\{1_{U\times\{h\}\times\phi_h(U)} : U \in \mathcal{B}\}$$

Theorem
$L_c(X) \rtimes_{\hat{\phi}} H \cong A_R(G)$

Examples

Graph groupoids

Let $E = (E^1, E^0, r, s)$ be a directed graph with associated graph groupoid

$$G_E = \{(\alpha x, |\alpha| - |\beta|, \beta x) \in \partial E \times \mathbb{Z} \times \partial E : x \in \partial E, \alpha, \beta \in E^*\}.$$

Recall that G_E is ample and all

$$Z(\mu, \nu) = \{(\mu x, |\mu| - |\nu|, \nu x) \in G_E : \mu x \in Z(\mu), \nu x \in Z(\nu)\}$$

and

$$Z(\mu, F, \nu) = \{(\mu x, |\mu| - |\nu|, \nu x) \in G_E : \mu x \in Z(\mu \backslash F), \nu x \in Z(\nu \backslash F)\},$$

is a basis of compact open bisections for a locally compact Hausdorff topology on G_E.
Therefore,

$$A_R(G_E) = \{1_{Z(\mu,F,\nu)} : \mu, \nu \in E^*, r(\mu) = r(\nu) \text{ and } F \subseteq r(\mu)E^1 \text{ is finite}\}$$

Leavitt path algebras as Steinberg algebras

Theorem
$L_R(E) \cong A_R(G_E)$

The isomorphism above is explicit, and is defined on generators as follows

$$\begin{aligned} v \in E^0: &\quad p_v \mapsto 1_{Z(v,v)} \\ e \in E^1: &\quad s_e \mapsto 1_{Z(e,r(e))} \\ e \in E^1: &\quad s_{e*} \mapsto 1_{Z(r(e),e)} \end{aligned}$$

Surjectivity follows from the observation

$$1_{Z(\mu,F,\nu)} = 1_{Z(\mu,r(\mu))} 1_{Z(r(\nu),\nu)} - \sum_{e \in F} 1_{Z(\mu e, r(\mu e))} 1_{Z(r(\nu e),\nu e)}.$$