Part 1: Introduction to groupoids

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What is a Groupoid?

A *groupoid* consists of a

- set G
- set $G^{(2)} \subset G \times G$, called composable pairs
- map $G^{(2)} \to G, (\gamma, \eta) \mapsto \gamma\eta$, called multiplication
- map $G \to G, \gamma \mapsto \gamma^{-1}$, called an inverse

satisfying the conditions

- involution: $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in G$.
- associativity: if $(\gamma, \eta), (\eta, \delta) \in G^{(2)}$ then $(\gamma\eta, \delta), (\gamma, \eta\delta) \in G^{(2)}$ and $(\gamma\eta)\delta = \gamma(\eta\delta)$,
- identities: $(\gamma, \gamma^{-1}) \in G^{(2)}$ for every $\gamma \in G$, and

  $(\gamma, \eta) \in G^{(2)} \Rightarrow (\gamma\eta)\eta^{-1} = \gamma$ and $\gamma^{-1}(\gamma\eta) = \eta$.

Groupoids

Every groupoid has

- a *range* map $r(\gamma) := \gamma\gamma^{-1}$.
- a *source* map $s(\gamma) := \gamma^{-1}\gamma$.
- a *unit space* $G^{(0)} := r(G) = s(G)$.

Lemma

Let G be a groupoid.We have

- $\gamma \in G \Rightarrow (r(\gamma), \gamma), (\gamma, s(\gamma)) \in G^{(2)}$ and $r(\gamma)\gamma = \gamma = \gamma s(\gamma)$
- $(\gamma, \eta) \in G^{(2)}$ if and only if $s(\gamma) = r(\eta)$; thus

$$G^{(2)} = \{(\gamma, \eta) \in G \times G : s(\gamma) = r(\eta)\}$$

- $r(u) = u = s(u)$ for all $u \in G^{(0)}$

Groupoids

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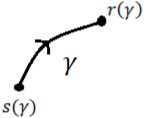
Visualizing groupoid elements



Groupoids

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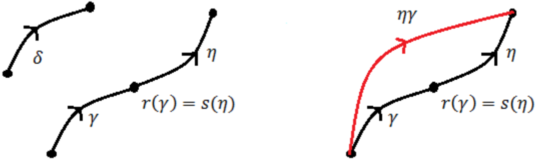
Visualizing multiplication



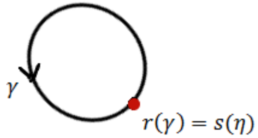
Structural notions

If $u \in G^{(0)}$, then

- the *isotropy group* at u is

$$G(u) := \{\gamma \in G : r(\gamma) = s(\gamma) = u\}$$

($G(u)$ is a bona fide group)



- the *orbit* of u is

$$[u] = \{v \in G^{(0)} : \exists \gamma \in G \text{ s.t. } r(\gamma) = v, s(\gamma) = u\}$$

Examples of groupoids

1. Groups: Every group $\mathcal{G}$ is a groupoid with the group operation and group inverses. In this case $\mathcal{G}^{(0)} = \{e\}$. A groupoid is a group if and only if its unit space is a singleton.

2. Matrix groupoids: Fix $N \in \mathbb{N}$. Let

$$R_N = \{1, 2, \ldots, N\} \times \{1, 2, \ldots, N\}.$$

Define

- $((i,j),(k,l)) \in R^{(2)}$ if $j = k$ and then $(i,j)(j,l) = (i,l)$, and
- $(i,j)^{-1} = (j,i)$.

Then R_N is a groupoid; it is a special cases of the following.

Examples of groupoids

3. Equivalence relations: Let X be a set, and $R \subset X \times X$ an equivalence relation.

Define

- $((x, y), (w, z)) \in R^{(2)}$ if $w = y$ and then $(x, y)(y, z) = (x, z)$, and
- $(x, y)^{-1} = (y, x)$.

Then R is a groupoid and

- $r(x, y) = (x, y)(x, y)^{-1} = (x, x)$ and
  $s(x, y) = (x, y)^{-1}(x, y) = (y, y)$,

  and therefore
- $R^{(0)} = \{(x, x) : x \in X\}$ (so we 'identify' $R^{(0)}$ with X).

Extreme cases:

- $R = X \times X$
- $R = \{(x, x) : x \in X\}$

Groupoids that are equivalence relations

- If G and H are groupoids, then $\phi : G \to H$ is a groupoid homomorphism if

$$(\gamma, \eta) \in G^{(2)} \Rightarrow (\phi(\gamma), \phi(\eta)) \in H^{(2)} \text{ and } \phi(\gamma\eta) = \phi(\gamma)\phi(\eta).$$

- If ϕ is bijective and ϕ^{-1} is also a groupoid homomorphism, the ϕ is a groupoid isomorphism.

- $R(G) = \{(r(\gamma), s(\gamma)) \in G^{(0)} \times G^{(0)} : \gamma \in G\}$ defines an equivalence relation on $G^{(0)}$, and $\gamma \mapsto (r(\gamma), s(\gamma))$ is a surjective groupoid homomorphism from G to R.

- We say G is principle if $\gamma \mapsto (r(\gamma), s(\gamma))$ is injective.

Lemma

G is algebraically isomorphic to an equivalence relation if and only if G is principle.

(More) Examples of groupoids

4. Group actions: Let a group H act on a set X by bijections.

Let
$$G = \{(x, g, y) \in X \times H \times X : x = g \cdot y\}.$$

Define

- $(x, g, y)(y, h, z) = (x, gh, z)$ (undefined otherwise)
- $(x, g, y)^{-1} = (y, g^{-1}, x)$

Then G is a groupoid and

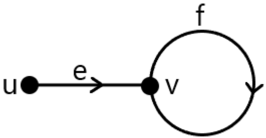
- $r(x, g, y) = (x, e, x)$ (e is the identity of H)
- $s(x, g, y) = (y, e, y)$, and
- $G^{(0)} = \{(x, e, x) : x \in X\}$ (we identify $G^{(0)}$ with X).
- G is sometimes called a transformation groupoid.
- G encodes the structural properties of (H, X), e.g. orbits and isotropy groups.

(More) Examples of groupoids

5. Directed graphs: Let $E = (E^1, E^0, r, s)$ be a directed graph; that is

- E^0 is a set vertices
- E^1 is a set of edges
- $r, s : E^1 \to E^0$ are the range and source maps

For example



Here $E^1 = \{e, f\}, E^0 = \{u, v\}, s(e) = u$ and $r(e) = v = r(f) = s(f)$.

(More) Examples of groupoids

5. Directed graphs: Let (E^1, E^0, r, s) be a directed graph. Then
   - a path of length n is a finite sequence of edges $e_1 e_2 \cdots e_n$ such that $r(e_i) = s(e_{i+1})$ for every $1 \leq i \leq n-1$. Let E^* denote all finite paths in E.
   - $v \in E^0$ is considered as path of length 0.
   - an infinite path is an infinite sequence $e_1 e_2 \cdots$ such that $r(e_i) = s(e_{i+1})$ for all $i \in \mathbb{N}$. Let E^∞ denote all infinite paths in E.
   - for $x \in E^*$, we let $xE^1 = \{e \in E^1 : s(e) = r(x)\}$.
   - the boundary path space is $\partial E = E^\infty \cup \{x \in E^* : xE^1 = \emptyset\} \cup \{x \in E^* : |xE^1| = \infty\}$.

(More) Examples of groupoids

5. Directed graphs: Let (E^1, E^0, r, s) be a directed graph
Define

$$G_E = \{(\alpha x, |\alpha| - |\beta|, \beta x) \in \partial E \times \mathbb{Z} \times \partial E : x \in \partial E, \alpha, \beta \in E^*\}$$

and

- multiplication by $(x, k, y)(y, l, z) = (x, k + l, z)$
- inversion by $(x, k, y)^{-1} = (y, -k, x)$.

Then

- G_E is a groupoid,
- $r(x, k, y) = (x, 0, x)$ and $s(x, k, y) = (y, 0, y)$,

  and
- $G_E^{(0)} = \{(x, 0, x) : x \in \partial E\}$ is identified with ∂E.

Topological groupoids

G is a topological groupoid if G is a groupoid with a locally compact topology such that

- $G^{(0)}$ is Hausdorff in the relative topology,
- $(\gamma, \eta) \mapsto \gamma\eta$ is continuous (w.r.t. relative product topology on $G^{(2)}$),and
- $\gamma \mapsto \gamma^{-1}$ is continuous.

Consequently, $r, s : G \to G^{(0)}$ are continuous.

• Many groupoids of interest are Hausdorff. However, there are many important examples of non-Hausdorff groupoids, making those attractive to study (c.f. REF).

• In this talk I will only consider Hausdorff groupoids.

Lemma

G is Hausdorff if and only if $G^{(0)}$ is closed in G.

Étale

- A function $f : X \to Y$ is a local homeomorphism if f is continuous and for every $x \in X$, there is an open neighborhood U of x such that $f(U) \subseteq Y$ is open and $f : U \to f(U)$ is a homeomorphism.

- A topological groupoid G is an étale groupoid if $r : G \to G^{(0)}$ is a local homeomorphism.

- G étale, then $B \subset G$ is a bisection if there is open $U \supseteq B$ such that $r : U \to r(U)$ and $s : U \to s(U)$ are homeomorphims onto open sets of $G^{(0)}$.

- G is an ample groupoid if it has a basis of **compact open** bisections.

Étale

Proposition

Let G be a Hausdorff groupoid. The following are equivalent.

- G is an étale groupoid
- $G^{(0)}$ is open in G
- r and s are local homeomorphism
- G has a basis of open bisections for its topology.

Examples of Topological groupoids

1. Groups
   - Every locally compact group $\mathcal{G}$ is a topological groupoid.
   - $\mathcal{G}$ is étale if and only if $\mathcal{G}$ is discrete (i.e. étale groupoids generalize discrete groups)

2. Equivalence relations:
   - If X is a locally compact space and $R \subset X \times X$ an equivalence relation. Then R is a topological groupoid with the relative product topology from $X \times X$.

Examples of Topological groupoids

3. Group actions:
   - If H is locally compact Hausdorff group acting on locally compact Hausdorff space X by homeomorphisms, then

     $$G = \{(x, g, y) \in X \times H \times X : x = g \cdot y\}.$$

     is a topological groupoid w.r.t. the relative product topology.
   - G is étale if and only if H is a discrete group. (so actually étale groupoids generalize discrete groups *actions*)
   - If X has a basis of compact open sets, then G is ample.

Examples of Topological groupoids

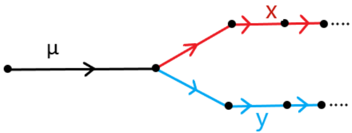
3. Directed graphs: Let $E = (E^1, E^0, r, s)$ be a directed graph. For $\mu \in E^*$ and $F \subseteq r(\mu)E^1$ finite, define the cylinder set

$$Z(\mu) = \{\mu x \in \partial E : x \in r(\mu)\partial E\}$$

and the punctured cylinder set

$$Z(\mu \backslash F) = Z(\mu) \backslash \cup_{e \in F} Z(\mu e).$$

For example



Here $Z(\mu) = \{\mu x, \mu y\}$ and $Z(\mu \backslash \{y\}) = \{\mu x\}$.
The cylinder and punctured cylinder sets form a basis of compact open sets for a locally compact Hausdorff topology on ∂E.

Examples of Topological groupoids

3. Directed graphs: Let $E = (E^1, E^0, r, s)$ be a directed graph.

Using the cylinder sets we can define a topology on G_E: for $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$ and $F \subseteq r(\mu)E^1$ finite we define

$$Z(\mu, \nu) = \{(\mu x, |\mu| - |\nu|, \nu x) \in G_E : \mu x \in Z(\mu), \nu x \in Z(\nu)\}$$

and

$$Z(\mu, F, \nu) = \{(\mu x, |\mu| - |\nu|, \nu x) \in G_E : \mu x \in Z(\mu \setminus F), \nu x \in Z(\nu \setminus F)\}.$$

These sets for a basis of compact open bisections for a locally compact Hausdorff topology on G_E, making G_E an ample groupoid.