ON GROUPOID ALGEBRAS WITH APPLICATIONS TO LEAVITT LABELLED PATH ALGEBRAS - TRAINING SESSION

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1. Leavitt path algebras as Steinberg algebras

We briefly recall some definitions seen during the first two talks. The exercise will be given at the end of the section. Let $E = (E^1, E^0, r, s)$ be a directed graph.

- For $x \in E^*$, we let $xE^1 = \{e \in E^1 : s(e) = r(x)\}$.
- The **boundary path space** is

$$\partial E = E^\infty \cup \{x \in E^* : xE^1 = \emptyset\} \cup \{x \in E^* : |xE^1| = \infty\}.$$

- $xE^1 = \emptyset$ means that $r(x)$ is a sink, $|xE^1| = \infty$ means that $r(x)$ is an infinite emitter.

For $\mu \in E^*$ and $F \subseteq r(\mu)E^1$ finite, define the **cylinder set**

$$Z(\mu) = \{\mu x \in \partial E : x \in r(\mu)\partial E\}$$

and the **punctured cylinder set**

$$Z(\mu \backslash F) = Z(\mu) \backslash \cup_{e \in F} Z(\mu e).$$

Define

$$G_E = \{(\alpha x, |\alpha| - |\beta|, \beta x) \in \partial E \times \mathbb{Z} \times \partial E : x \in \partial E, \alpha, \beta \in E^*\}$$

and

- multiplication by $(x, k, y)(y, l, z) = (x, k + l, z)$
- inversion by $(x, k, y)^{-1} = (y, -k, x)$.

Then

- G_E is a groupoid,
- $r(x, k, y) = (x, 0, x)$ and $s(x, k, y) = (y, 0, y)$, and
- $G_E^{(0)} = \{(x, 0, x) : x \in \partial E\}$ is identified with ∂E.

Using the cylinder sets we can define a topology on G_E: for $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$ and $F \subseteq r(\mu)E^1$ finite we define

$$Z(\mu, \nu) = \{(\mu x, |\mu| - |\nu|, \nu x) \in G_E : \mu x \in Z(\mu), \nu x \in Z(\nu)\}$$

and

$$Z(\mu, F, \nu) = \{(\mu x, |\mu| - |\nu|, \nu x) \in G_E : \mu x \in Z(\mu \backslash F), \nu x \in Z(\nu \backslash F)\}.$$

These sets form a basis of compact open bisections for a locally compact Hausdorff topology on G_E, making G_E an ample groupoid.

Define

$$A_R(G) = \{f : G \to R : f \text{ is continuous with compact support}\}.$$

We give $A_R(G)$ algebraic structure: define

- addition pointwise, then $A_R(G)$ becomes an R-module
- multiplication by

$$f * g(\gamma) = \sum_{r(\eta)=r(\gamma)} f(\eta)g(\eta^{-1}\gamma)$$

  (called a **convolution product**)

Then $A_R(G)$ is an R-algebra called the **Steinberg algebra** associated with G.

Since a Steinberg algebra is spanned by characteristic functions on compact open bisections, we have that

$$A_R(G_E) = \mathrm{span}_R\{1_{Z(\mu,F,\nu)} : \mu,\nu \in E^*, r(\mu) = r(\nu) \text{ and } F \subseteq r(\mu)E^1 \text{ is finite}\}.$$

Exercise 1. In $A_R(G_E)$, consider the functions as follows:

$$\begin{array}{ll} \text{for } v \in E^0: & p_v = 1_{Z(v,v)} \\ \text{for } e \in E^1: & s_e = 1_{Z(e,s(e))} \\ \text{for } e \in E^1: & s_{e*} = 1_{Z(s(e),e)} \end{array}$$

Prove that the following hold in $A_R(G_E)$

- (V) for $v, w \in E^0$, $p_v * p_v = p_v$ and $p_v * p_w = 0$ if $v \neq w$,
- (E1) for $e \in E^1$, $p_{s(e)} * s_e = s_e$ and $s_e * p_{r(e)} = s_e$,
- (E2) for $e \in E^1$, $p_{r(e)} * s_e^* = s_e^*$ and $s_e^* * p_{r(e)} = s_e^*$ (it is analogous to (E1)),
- (CK1) for $e, f \in E^1$, $s_e^* * s_e = p_{r(e)}$ and $s_e^* s_f = 0$ if $e \neq f$,
- (CK2) for $v \in E^0$ such that $0 < |vE^1| < \infty$ (that is, v is not a sink nor an infinite emitter),
$$p_v = \sum_{e \in s^{-1}(v)} s_e * s_e^*.$$

2. A dynamical point of view for the LPA relations

Exercise 2. For a given set X, let $\mathcal{I}(X) = \{f : A \to B \mid A, B \subseteq X \text{ and } f \text{ is a bijection}\}$ (the empty function is a bijection!).

(a) For $f : A \to B, g : C \to D \in \mathcal{I}(X)$, let $g \circ f : f^{-1}(B \cap C) \to g(B \cap C)$ be given by $(g \circ f)(x) = g(f(x))$, where $x \in X$. Prove that $g \circ f \in \mathcal{I}(X)$.

(b) For $f : A \to B, g : C \to D \in \mathcal{I}(X)$ such that $A \cap C = \emptyset = B \cap D$, show that there is natural way to define $f \cup g \in \mathcal{I}(X)$ (one can actually define $f \cup g$ whenever $f \circ g^{-1}$ and $f^{-1} \circ g$ are identity maps or the empty function).

Back to graphs, let $\partial E^{\geq 1} = \{\mu \in \partial E : |\mu| \geq 1\}$. We define a map $\sigma : \partial E^{\geq 1} = \partial E$ as follows

$$\begin{cases} \sigma(e) = r(e), & \text{if } e \in E^1 \cap \partial E \\ \sigma(e\mu) = \mu, & \text{if } e \in E^1 \text{ and } \mu \in \partial E^{\geq 1}. \end{cases}$$

The map σ is called the **shift map**. This gives a partially defined dynamics on ∂E.

Exercise 3. Consider the following functions:

- $P_v = Id_{Z(v)} : Z(v) \to Z(v)$ for $v \in E^0$,
- $S_e : Z(e) \to Z(r(e))$ given by $S_e(\nu) = \sigma(\nu)$ if $\nu \in Z(e)$, where $e \in E^1$.

(a) Prove that $P_v \in \mathcal{I}(\partial E)$ and $S_e \in \mathcal{I}(\partial E)$ for every $v \in E^0$ and $e \in E^1$. Describe S_e^{-1}

(b) Prove that the following holds in $\mathcal{I}(\partial E)$
- (V) for $v, w \in E^0$, $P_v \circ P_v = P_v$ and $P_v \circ P_w = \emptyset$ if $v \neq w$,
- (E1) for $e \in E^1$, $P_{s(e)} \circ S_e = S_e$ and $S_e \circ P_{r(e)} = S_e$,
- (E2) for $e \in E^1$, $P_{r(e)} \circ S_e^{-1} = S_e^{-1}$ and $S_e^{-1} \circ P_{r(e)} = S_e^{-1}$,
- (CK1) for $e, f \in E^1$, $S_e^{-1} \circ S_e = P_{r(e)}$ and $S_e^{-1} \circ S_f = \emptyset$ if $e \neq f$,
- (CK2) for $v \in E^0$ such that $0 < |vE^1| < \infty$ (that is, v is not a sink nor an infinite emitter),
$$P_v = \bigcup_{e \in s^{-1}(v)} S_e \circ S_e^{-1}.$$

(c) Prove that each S_e is a homeomorphism, so that σ is a local homeomorphism.