INTRODUCTION TO GRAPH ALGEBRAS AND ATTEMPTS AT THEIR CLASSIFICATION

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Abstract. After a short review of the K_0-group concept, we introduce some classes of algebras related to a directed graph and present a hands-on method of computing their (pointed) K_0-groups. We put most of our focus on Leavitt path algebras, but the methods we present can also be used for some other graph algebras.

The examples we present illustrate that the K_0 group does not classify Leavitt path algebras. However, if one considers the grading of these algebras and adjusts the definition of the K_0-group to reflect the existence of this grading, the situation becomes more interesting. The Graded Classification Conjecture states that this adjusted version of the (pointed) K_0-group is a complete invariant of Leavitt path algebras over a field. After presenting some examples illustrating the conjecture, we discuss the context in which this conjecture has been formulated, its current status and, its relations with another conjecture.

Towards the end, we shift gears to representations of Leavitt path algebras. We present a class of irreducible representations which classify all irreducible representations up to the equality of their kernels and then we do the same for the graded irreducible representations.

Some necessary background in algebra will be reviewed as we progress with the material.

Overview. This short course consists of the following five lectures each complemented with many examples and exercises.

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Lectures 1 and 2. Introduction to Grothendieck groups and graph algebras

Background on Grothendieck group. Before we start looking at some graph algebras, let us address the following two general questions.

(1) What is K_0?
(2) When does K_0 classify a class of rings (or algebras)?

Let R be a ring and consider the class of **free** left modules (the nicest, the best behaved R-modules). If you are not sure what a free module is, imagine a module which has a *basis* – a set of linearly independent vectors which generate the entire module. So, this basis has the same properties as a *vector space* basis. In fact, if R is a field, free modules are *exactly* the vector spaces and all the modules are free.

A direct summand of a free module is said to be a **projective** module.

Free and projective

Projective modules can be introduced also categorically. Namely, the following three conditions are equivalent for an R-module P. (1) P is a direct summand of a free module; (2) For any diagram of R-modules,

$$\begin{tikzcd} & P \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \quad \text{there is } f \text{ such that} \quad \begin{tikzcd} & P \arrow[d] \arrow[dl, "f"'] & \\ A \arrow[r] & B \arrow[r] & 0; \end{tikzcd}$$

(3) Every short exact sequence of R-modules $\quad 0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0 \quad$ splits.

Exercise 1. Show that the above three requirements are equivalent. This is not crucial for understanding the rest of the material, but it is a good exercise for you. For (3) $\Rightarrow$ (1) you can use that for any R-module M, there is a free module F and a surjective homomorphism $F \to M$.

Solution to Exercise 1. (1) $\Rightarrow$ (2) If π denotes the natural projection $F \to P$ and ι the natural injection $P \to F$, show first that for $\begin{tikzcd} F \arrow[r, "\pi"] & P \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$ there is g such that $\begin{tikzcd} F \arrow[r, "\pi"] \arrow[d, "g"] & P \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0. \end{tikzcd}$

Then define $f(p)$ for $p \in P$ by $g(\iota(p))$.

(2) $\Rightarrow$ (3) Consider $\begin{tikzcd} & P \arrow[d, "="] & \\ B \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$.

(3) $\Rightarrow$ (1) Let F be a free module such that $\pi : F \to P$ is a surjection. By (3), $0 \to \ker \pi \to F \to P \to 0$ splits.

Example 1. If K is a field, the direct sum $K \oplus K \ldots \oplus K = K^n$ is a free module and the "standard basis" is a basis of K^n. There are no nontrivial projective K-modules which are not free.

Example 2. To contrast the previous example, we exhibit a ring and its projective module which is not free. Consider the ring $\mathbb{Z}_6$ and note that it is a free module over itself ($\{1\}$ is a basis). There is a module isomorphism $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$, so $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are projective. However, neither is free because their nonzero elements are not linearly independent (e.g., for $2 \in \mathbb{Z}_6$ and $1 \in \mathbb{Z}_2$, $2 \cdot 1 = 0$ in $\mathbb{Z}_2$), so they do not contain any basis.

Let us go back to the first example and consider the *dimensions* $1, 2, 3 \ldots$ of the free modules $K, K^2, K^3, \ldots$. One would want these dimensions to be elements of a group, think of it as the *group of dimensions*. So, to have the identity and the inverses of these dimensions, we "add" them to the list of dimensions and end up with

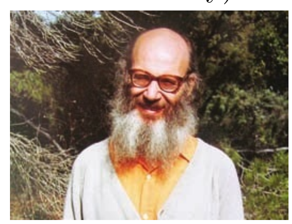
$$\ldots,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots$$

And this is it: the Grothendieck group $K_0(K)$ is the group of integers $\mathbb{Z}$.

The Grothendieck group of a ring. The general construction resembles this example.

(1) Consider the isomorphism class $[P]$ of a finitely generated projective module P. Set of all such classes is a **monoid** (a set with an associative operation with the identity) under

$$[P] + [Q] = [P \oplus Q].$$



This monoid is usually denoted by $\mathcal{V}(R)$.

(2) Force the cancellativity to hold on such a monoid, and

(3) Complete to a group.

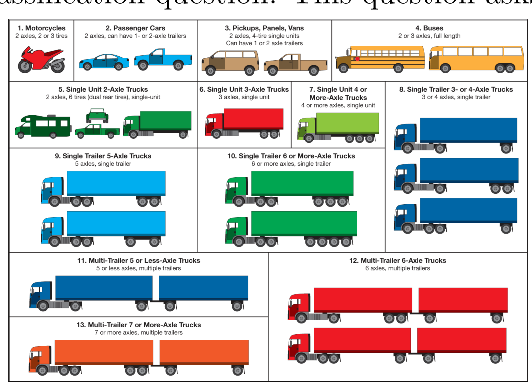
The steps (2) and (3) can be achieved by considering the free abelian group on generators $[P] \in \mathcal{V}(R)$ subject to the relations of the form $[P] + [Q] = [P \oplus Q]$. The resulting group is called **the Grothendieck group** of R and it is denoted by $K_0(R)$. The elements of $K_0(R)$ can be written as $[P] - [Q]$ for $[P], [Q] \in \mathcal{V}(R)$. Note that $\mathcal{V}(R)$ and $K_0(R)$ are *abelian* since $P \oplus Q \cong Q \oplus P$.

There are alternative constructions of $K_0(R)$. One of them resembles the classical construction of $\mathbb{Z}$ from $\mathbb{N}$ even more than the approach we presented. By yet another approach, $K_0(R)$ can be introduced using the idempotents, not finitely generated projective modules.

Example 3. We have seen that if K is a field, $\mathcal{V}(K) \cong \mathbb{N} \cup \{0\}$. We use $\mathbb{Z}^+$ to denote $\mathbb{N} \cup \{0\}$. So, $K_0(K) \cong \mathbb{Z}$. For a less trivial example, let R be a ring such that $R \oplus R \cong R$. We will see a graph algebra which has this property. So, we have that $[R] = [R] + [R]$ holds in $\mathcal{V}(R)$. Imposing the cancellativity, we have that $[R] = 0$ holds in $K_0(R)$. Hence, $[F]$ is 0 in $K_0(R)$ for every free module and, as a consequence, $[P]$ is 0 for every projective module. Thus, $K_0(R) = 0$.

Classification question. After getting to know some graph algebras, we shall see many other examples of K_0-groups. Before that, let us state the classification question. This question asks how well K_0 reflects the properties of a class of rings or algebras.

$R \cong S$ as rings (or algebras) if and only if $K_0(R) \cong K_0(S)$ as (pointed) groups?



We will talk more about "pointed" later. Of course, the direction ($\Rightarrow$) always holds. The direction ($\Leftarrow$) rarely holds: it holds only for some very "nice" algebras, for example matrix algebras over a field.

Path algebras. Before defining a path algebra, we review the concept of *algebra* and of a *directed graph*. Recall that A is an **algebra** over a field K if A is a K-vector space with an operation $\cdot$ which makes $(A, +, \cdot)$ into a ring and such that $k(a \cdot b) = ka \cdot b = a \cdot kb$.

A directed graph E consists of a set of vertices E^0, a set of edges E^1, and the source and the range maps $\mathbf{s}$ and $\mathbf{r}$ defined on E^1.

$$\bullet_{\mathbf{s}(e)} \xrightarrow{e} \bullet_{\mathbf{r}(e)}$$

A **path** of a directed graph E is a sequence $e_1 \ldots e_n$ of edges such that

the range of e_i is the source of e_{i+1}

for $i = 1, \ldots, n-1$.

$$\bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \bullet \quad \ldots \quad \bullet \xrightarrow{e_n} \bullet$$

Such path has the length n. We think of a vertex $\bullet$ as a path of length zero.

The paths can be multiplied by **concatenation**:

if the range of p is the source of q then their product is the path pq, otherwise it is zero.

If K is a field, form a K-vector space with the paths as the basis and extend the concatenation multiplication to this set so it is distributive and $k(p \cdot q) = kp \cdot q = p \cdot kq$ for $k \in K$ and paths p, q. This is the **path algebra** $P_K(E)$.

Example 4. If E is $\bullet_u \xrightarrow{e} \bullet_v \xrightarrow{f} \bullet_w$ the set of paths is $\{u, \quad v, \quad w, \quad e, \quad f, \quad ef\}$.

Some of the products are $\quad we = 0, \quad ue = e, \quad fe = 0 \quad$ and

the product of e and f is the basis element ef.

An element of $P_{\mathbb{C}}(E)$ is a $\mathbb{C}$-linear combination of these six paths. For example, $3e + \sqrt{5}\,ef$, and $(2+i)v - \frac{3}{4}f$ are some of the elements of $P_{\mathbb{C}}(E)$. In particular, the sum of e and f, for example, is the linear combination $e + f$.

Note also that this algebra is isomorphic to the algebra $\mathbb{T}_3(K)$ of the upper triangular matrices with entries in K. Indeed, if e_{ij} denotes the **standard matrix unit** with 1 on (i,j) spot and 0 elsewhere, we have that the map

$$u \mapsto e_{11}, \quad e \mapsto e_{12}, \quad ef \mapsto e_{13}, \quad v \mapsto e_{22}, \quad f \mapsto e_{23}, \text{ and } w \mapsto e_{33}$$

extends to an isomorphism $P_K(E) \to \mathbb{T}_3(K)$. We represent this map by $\begin{bmatrix} u & e & ef \\ 0 & v & f \\ 0 & 0 & w \end{bmatrix}$

Exercise 2. Determine the path algebras of the following graphs by writing their basis set.

$$\bullet_u \xrightarrow{e} \bullet_v \circlearrowleft f \qquad\qquad \bullet_u \xleftarrow{e} \bullet_v \circlearrowleft f$$

Solutions to Exercise 2. The set $\{u, v, e, f, ef, ff, eff, fff, \ldots, ef^{n-1}, f^n \ldots\}$ is a basis of the first path algebra. The set $\{u, v, e, f, fe, ff, ffe, fff, \ldots f^{n-1}e, f^n, \ldots\}$ is a basis of the second path algebra.

Before the next exercise, recall that a graph is *finite* if it has finitely many vertices and edges.

Exercise 3. If the path algebra $P_K(E)$ of E has a finite dimension over K, then E is finite and there are no cycles. Note that the converse holds also but an argument for that is more involved.

Solutions to Exercise 3. Assuming that E is not finite implies that it either has infinitely many vertices or infinitely many edges. In either case, those create an infinite subset of the basis, so $P_K(E)$ is not finite dimensional. If there is a cycle c, then $c, cc, ccc\ldots$ are basis elements, so $P_K(E)$ is not finite dimensional.

Another way to introduce the path algebra. $P_K(E)$ is a **free K-algebra** with vertices and edges as **generators** subject to the two axioms below. For any $v, w \in E^0$ and $e \in E^1$,

V $vw = 0$ if $v \neq w$ and $vv = v$, and
E1 $\mathbf{s}(e)e = e\mathbf{r}(e) = e$.

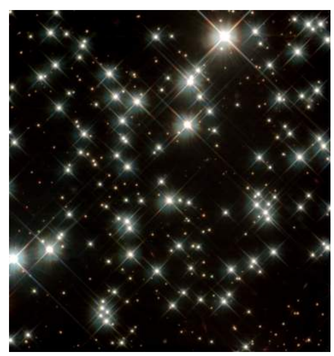
The fact that $P_K(E)$ is a free K-algebra implies that its elements are linear combinations of *products* of the basis vectors. Showing that the two definitions are equivalent is a great exercise in understanding the difference between a basis of a K-vector space and of a free K-algebra.

Exercise 4. Show that the two definitions of a path algebra are equivalent.

Solutions of Exercise 4. Let P_1 denotes the $P_K(E)$ defined using the first definition and P_2 denote the same algebra using the second definition.

The K-algebra basis of P_2 is $E^0 \cup E^1$. Since $e_i = e_i\mathbf{r}(e_i)$, $e_{i+1} = \mathbf{s}(e_{i+1})e_{i+1} = \mathbf{r}(e_i)e_{i+1}$ for $i = 0, \ldots, n-1$, any path $e_1 \ldots e_n$ is a product of the K-algebra basis elements. Thus, the set of all paths of E is contained in P_2 and so $P_1 \subseteq P_2$. Conversely, the basis of P_2 is contained in P_1 and the axioms V and E1 hold in P_1. This implies that $P_2 \subseteq P_1$.

Adding more structure. First, an involution. Path algebra is nice, but this algebra fails to capture a lot of info on the graph. We are adding more structure. In particular, the $*$ in "C^*-algebra" refers to an unary operation called an involution.



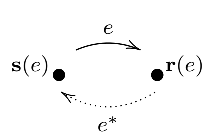
A ring R is a ring with **involution** or a $*$-ring, if there is an operation $* : R \to R$ such that, for any $x, y \in R$,

$(x + y)^* = x^* + y^*$,
$(xy)^* = y^*x^*$, and
$(x^*)^* = x$.

For more on $*$-rings see [6].

Example 5. (1) The identity on any commutative ring is an involution.
(2) The map $a + ib \mapsto a - ib$ is an involution on the ring of complex numbers $\mathbb{C}$.
(3) If R is a commutative ring and $\mathbb{M}_n(R)$ the ring of $n \times n$ matrices, then the transpose of matrices is an involution.
(4) If a ring R has an involution $r \mapsto r^*$, the matrix ring $\mathbb{M}_n(R)$ has involution given by
$$(r_{ij}) \mapsto (r^*_{ji})$$

Adding the ghost paths. The **extended graph** $\widehat{E}$ of a graph E is the graph with vertices E^0, edges $E^1 \cup \{e^* \mid e \in E^1\}$, and with the range and the source maps the same on E^1 and given by $\mathbf{s}(e^*) = \mathbf{r}(e)$ and $\mathbf{r}(e^*) = \mathbf{s}(e)$ on the added edges, called the **ghost edges**.



Consideration of the extended graph is a way to impose a natural involution to an algebra defined by a graph. To make sure that the source and range maps make sense on the added edges, we add the following "ghost version" of E1.

E2 $\quad \mathbf{r}(e)e^* = e^*\mathbf{s}(e) = e^*$

We extend $*$ to all paths by: $v^* = v$ for $v \in E^0$ and by $p^* = e_n^* \dots e_1^*$ for a path $p = e_1 \dots e_n$. We say that p^* is a **ghost path.**

Example 6. In the example with E being

$$\bullet_u \xrightarrow{e} \bullet_v \xrightarrow{f} \bullet_w$$

some of the "obvious" products are $\quad e^*f = 0, \quad f^*u = 0, \quad ef^* = 0, \quad ue^* = 0.$

There are some "not so obvious" products in this example. For example, what are e^*e and f^*f (if anything) and what are ee^* and ff^* (if anything)? Here, by "if anything" we mean to imply the possibility that these elements are not identified with any other basis elements of the path algebra of the extended graph. To understand the answers, we briefly digress to...

Projections and partial isometries. In a $*$-ring, an idempotent ($pp = p$) which is self-adjoint ($p^* = p$) is called a **projection**. For example, the vertices of the path algebra of the extended graph are projections.

An element x of a $*$-ring is a **partial isometry** if $xx^*x = x$.

If x is a partial isometry, then $p = xx^*$ and $q = x^*x$ are projections and

$$px = x \text{ and } xq = x.$$

Compare these relations with $\mathbf{s}(e)e = e$ and $e\mathbf{r}(e) = e$. Because of this analogy, one can think of p as "the source" and q as "the range" of x. (Caution: the established $*$-terminology is "final" for "source" and "initial" for "range".)

This indicates that one wants edges to be partial isometries in which case e^*e can indeed be thought of as the range of e and ee^* as a kind of a source of e. These considerations lead us to the following two axioms called CK1 and CK2 where CK is short for Cuntz-Krieger.

CK1 axiom. To make edges be partial isometries, one requires that

$$e^*e = \mathbf{r}(e)$$

since then $ee^*e = e\mathbf{r}(e) = e$ by E1.

Another requirement is that we want the "sources" ee^* and ff^* to be **mutually orthogonal** (in the sense that their product is commutative and it is zero) for edges $e \neq f$. This ends up being equivalent by requiring that $e^*f = 0$ for $e \neq f$. The two requirements are combined in the following axiom.

CK1 $\quad e^*e = \mathbf{r}(e)$ and $e^*f = 0$ if $e \neq f$.

Exercise 5. Show that e^* is also a partial isometry for every edge e and that the condition $ee^*ff^* = 0$ is equivalent with the condition $e^*f = 0$ for any $e, f \in E^1$ with $e \neq f$.

Solutions to Exercise 5. $e^*ee^* = \mathbf{r}(e)e^* = e^*$ by E2. If $e^*f = 0$, then $ee^*ff^* = 0$. For the converse, note that

$$e^*f = e^*ee^*ff^*f = e^*(ee^*)(ff^*)f = e^*0f = 0.$$

CK2 axiom. We would like to keep track of the number of other edges the source $\mathbf{s}(e)$ of an edge e emits. So, if a vertex does not emit any edges (such a vertex is called a **sink**), we impose no condition on it. Also, if a vertex emits infinitely many edges (such a vertex is called an **infinite emitter**), we also impose no condition on it. We say that a vertex v is **regular** if the set $\mathbf{s}^{-1}(v)$ of edges v emits is *nonempty and finite*. Alternatively, you can say that a vertex is regular if it is neither a sink nor an infinite emitter. So, we require that the following holds for every *regular* vertex v.

CK2 $\quad v = \sum ee^* \quad$ where the sum is taken over $e \in \mathbf{s}^{-1}(v)$.

Example 7. For example, for the graph $\bullet_u \xrightarrow{e} \bullet_v \xrightarrow{f} \bullet_w$ we have that

$$e^*e = v, \quad f^*f = w, \quad ee^* = u, \quad ff^* = v.$$

For the graph $\bullet_u \xleftarrow{e} \bullet_v \xrightarrow{f} \bullet_w$ we have that $\quad v = ee^* + ff^* \quad$ indicating that v emits exactly two edges. Note that $ee^* \neq v$ and $ff^* \neq v$.

We got ourselves some graph algebras. If K is a field, and E a graph, the **Leavitt path algebra** $L_K(E)$ of E over K is the quotient of the path algebra $P_K(\widehat{E})$ over the extended graph $\widehat{E}$ with respect to the (ideal generated by the) relations CK1 and CK2.

By an alternative definition, $L_K(E)$ is a free K-algebra on vertices, edges and ghost edges subject to the relations

V, E1, E2, CK1, and CK2.

Before we go to examples, let us introduce another graph algebra. Let us digress to $*$-rings for a moment to introduce an order on the set of projections given by

$$p \leq q \quad \text{if} \quad p = pq = qp.$$

Convince yourself that this relation is indeed an order (reflexive, antisymmetric and transitive). The following requirement is denoted by CK3.

CK3 $ee^* \leq \mathbf{s}(e)$ for every edge e.

Exercise 6. Show that CK3 holds in $L_K(E)$.

Solutions for Exercise 6. For any edge e, $\mathbf{s}(e)ee^* = ee^*$ by E1 and $ee^*\mathbf{s}(e) = ee^*$ by E2.

Let K be the field of complex numbers $\mathbb{C}$ now. The **graph C*-algebra** $C^*(E)$ of E is the universal C^*-algebra generated by vertices and edges such that vertices are mutually orthogonal projections, that edges are partial isometries, and that CK1, CK2, and CK3 hold.

The previous exercise shows that one does not have to require CK3 for $L_K(E)$ since it follows from E1 and E2. Conversely, the following holds.

Exercise 7. Show that E1 and E2 hold in $C^*(E)$.

Solution to Exercise 7. For an edge e, $e\mathbf{r}(e) = ee^*e = e$ and $\mathbf{s}(e)e = \mathbf{s}(e)ee^*e = (\mathbf{s}(e)ee^*)e = ee^*e = e$ where the one-to-last equality holds by CK3. E2 follows from "starring" the two relations of E1.

The interest in graph C^*-algebras stems from the fact that some of them can represent C^*-algebras which are traditionally not related to any graphs and that such representation may simplify the study of such "non-graph" C^*-algebras. In addition, some invariants (e.g. K-theory) can be computed more easily for graph than for "non-graph" C^*-algebras.

The algebra $L_{\mathbb{C}}(E)$ is a pre-normed space and $C^*(E)$ is the norm *completion* of $L_{\mathbb{C}}(E)$. Because of this, the Leavitt path algebras can be considered as algebraic simplifications of the graph C^*-algebras.

$$\begin{array}{ccc} \text{Graph } C^*\text{-algebras} & \leftrightsquigarrow & \text{Leavitt Path Algebras} \\ \text{in} & & \text{in} \\ \text{Operator Theory world} & \leftrightsquigarrow & \text{Algebra world} \end{array}$$

In the interest of time and to ensure that our focus is not spread out to too many topics, we will not go into the precise definition of a C^*-algebra and will concentrate on Leavitt path algebras only. We want to emphasize, though, that most of the material we present can be applied to the graph C^*-algebras also.

Let us consider some examples of Leavitt path algebras next.

Example 8. Let us return to the example with E being $\bullet_u \xrightarrow{e} \bullet_v \xrightarrow{f} \bullet_w$.

Recall that $P_K(E)$ is generated by six generators: the three vertices, two edges and ef. When we add the two ghost edges e^* and f^* and the ghost path f^*e^* to this list, we have nine generators of $L_K(E)$. These generators and their products are related as in Example 7.

Recall that $P_K(E)$ is isomorphic to $\mathbb{T}_3(K)$ by $\begin{bmatrix} u & e & ef \\ 0 & v & f \\ 0 & 0 & w \end{bmatrix}$. Consider the involution on the 3×3 matrices over K given by taking the transpose and possibly involuting its entries based on any possible involution on K. To extend the isomorphism $P_K(E) \cong \mathbb{T}_3(K)$ to agree with $*$ (in the sense that $f(x^*) = f(x)^*$), we have that e^* should correspond to $e_{12}^* = e_{21}$. Similarly, f^* should correspond to $e_{23}^* = e_{32}$ and $(ef)^*$ to $e_{13}^* = e_{31}$. Hence,

$L_K(E)$ is isomorphic to the **matrix** algebra
$\mathbb{M}_3(K)$
by the isomorphism which we can represent by
$$\begin{bmatrix} u & e & ef \\ e^* & v & f \\ f^*e^* & f^* & w \end{bmatrix}.$$

Similarly, if E is

$$\bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet \cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

then $L_K(E)$ is isomorphic to $\mathbb{M}_n(K)$ via $\begin{array}{ccc} v_i & \leftrightarrow & e_{ii} \\ e_i & \leftrightarrow & e_{i\,i+1}. \end{array}$

Example 9. If E is $\bullet^v \circlearrowright e$, then $P_K(E)$ is generated by $v, e, ee, eee, \ldots$ and $L_K(E)$ by these elements and $e^*, e^*e^*, e^*e^*e^*, \ldots$.

By E1 and E2, v is a neutral element for all the generators and $e^*e = v$ by CK1 and $ee^* = v$ by CK2. So, v is the *identity* and e and e^* are invertible and mutually inverse to each other.

It turns out that $L_K(E)$ of this graph can be realized also as an algebra not related to any graph, the algebra $K[x,x^{-1}]$ of **Laurent polynomials** over the field K.

$$K[x,x^{-1}] = \{\textstyle\sum_{i=-m}^{n} k_i x^i \mid m,n \text{ are nonnegative integers and } k_i \in K\}.$$

The isomorphism is induced by $\begin{array}{c} v \leftrightarrow 1 \\ e \leftrightarrow x \end{array}$ (thus $e^* \leftrightarrow x^{-1}$).

Before proceeding with more examples, let us pause to note that the (one and only) vertex in the previous example is the identity of the algebra. In the example prior to that, the sum of three vertices is the identity. More generally, the following holds.

Exercise 8. If E has finitely many vertices, then *their sum is the identity* of $L_K(E)$. If E is any graph, then the finite sums of vertices are the **local units** in the sense that for every $x, y \in L_K(E)$, there is a local unit u such that $xu = ux = x$ and $uy = yu = y$.

Solution to Exercise 8. For a solution of this exercise, try to convince yourself that the axioms of $L_K(E)$ imply that every element x of $L_K(E)$ can be written as a sum $\sum_{i=1}^{n} k_i p_i q_i^*$ for some n, paths p_i and q_i, and elements $k_i \in K$, for $i = 1, \ldots, n$. Indeed, by definition, an element of $L_K(E)$ is a K-linear combination of finite products of paths and ghost paths. CK1 ensures that the ghost paths can be pushed to the back. Hence, we obtain the desired form. We note that [1] contains more details on this and other properties of Leavitt path algebras.

The claim then follows since the sum u of different vertices appearing as sources of paths p_i and q_i is such that $xu = ux = x$ by axioms V, E1, and E2.

Examples 8 and 9 extend to any finite graphs in which no cycle has an *exit* (i.e. an edge which is not in the cycle but which originates at a vertex of the cycle). If E is such a graph, $L_K(E)$ is isomorphic to a direct sum of matrix algebras over K and $K[x,x^{-1}]$. This is because every path in a finite graph with cycles which have no exits leads to a "dead end" – either a sink or to a cycle which one cannot exit after entering. Say that there are n sinks $v_1, \ldots, v_n$ and there are k_i paths p_{ij}, $j = 1, \ldots, k_i$, $i = 1, \ldots, n$, ending in v_i and m cycles $c_1, \ldots, c_m$ and l_i paths q_{ij}, $j = 1, \ldots, l_i$, $i = 1, \ldots, m$, ending in a fixed arbitrary vertex of c_i, such that q_{ij} does not contain c_i. Then

$$L_K(E) \cong \bigoplus_{i=1}^{n} \mathbb{M}_{k_i}(K) \oplus \bigoplus_{i=1}^{m} \mathbb{M}_{l_i}(K[x,x^{-1}])$$

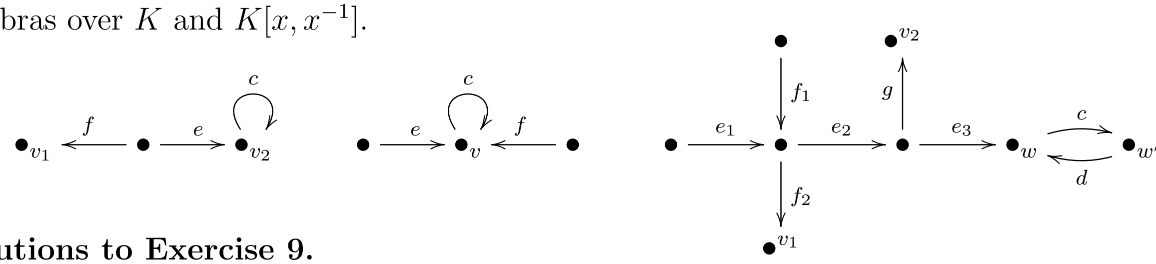
by $p_{ij} p_{il}^* \leftrightarrow e_{jl} \in \mathbb{M}_{k_i}(K)$ for $j, l = 1, \ldots, k_i$, $i = 1, \ldots, n$, and $q_{ij} c_i^k q_{il}^* \leftrightarrow x^k e_{jl} \in \mathbb{M}_{l_i}(K[x,x^{-1}])$ for $j, l = 1, \ldots, l_i$, $i = 1, \ldots, m$, $k \in \mathbb{Z}$.

Example 10. If E is $\bullet^v \xrightarrow{e} \bullet^w \circlearrowright f$, then there are two paths, w and e, which end at w without counting the loop f. Hence, $L_K(E) \cong \mathbb{M}_2(K[x,x^{-1}])$.

If F is $\bullet^v \underset{f}{\overset{e}{\rightleftarrows}} \bullet^w$, let us consider the cycle as ef in which case v is its base. There are two paths, v and f, ending in v which do not contain entire cycle ef. So, $L_K(F) \cong M_2(K[x,x^{-1}])$. When considering the cycle as fe and w as its base, we also have two paths (w and e in this case) which end at the selected vertex (w in this case), so we also end up with $L_K(F) \cong M_2(K[x,x^{-1}])$.

These examples also illustrate that two very different graphs (E has a source while F does not, for example) can have isomorphic Leavitt path algebras. As we shall soon see, the graded structure introduces a distinguishing feature between algebras over graphs such as E and F.

Exercise 9. Identify the Leavitt path algebras of the following graphs as sums of matrix algebras over K and $K[x, x^{-1}]$.



Solutions to Exercise 9.

(1) The first graph has one sink and one cycle. There are two paths, v_1 and e, ending in v_1 and two paths, v_2 and f, ending in v_2. Thus, $L_K(E) \cong \mathbb{M}_2(K) \oplus \mathbb{M}_2(K[x, x^{-1}])$.
(2) The second graph has no sinks and a single cycle. There are three paths v, e and f, ending in v so $L_K(E) \cong \mathbb{M}_3(K[x, x^{-1}])$.
(3) The third graph has two sinks v_1 and v_2 and one cycle cd. There are four paths v_1, f_2, f_1f_2, and e_1f_2 ending in v_1, and five paths, v_2, g, e_2g, e_1e_2g, and f_1e_2g ending in v_2. Choosing w for a base of the cycle cd, there are six paths $w, e_3, e_2e_3, e_1e_2e_3$, $f_1e_2e_3$, and d ending in w. Choosing w' instead of w for a base of cd would lead to the same number. Thus, $L_K(E) \cong \mathbb{M}_4(K) \oplus \mathbb{M}_5(K) \oplus \mathbb{M}_6(K[x, x^{-1}])$.

Not every Leavitt path algebra is isomorphic to a sum of matricial algebras over commutative rings as the next example shows. This example is one of the most famous examples of Leavitt path algebras as well as one of the forces motivating the definition of this class of algebras.

Example 11. Consider E to be the graph  and let us denote the vertex by 1 since (now we know) it is the identity of $L_K(E)$. Note that the relations on the generators include $e^*e = f^*f = 1$ and $ee^* + ff^* = 1$.

These relations imply that the pair of maps

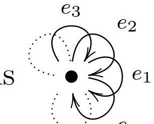
$$x \mapsto (e^*x, f^*x) \text{ and } (x, y) \mapsto ex + fy$$

are mutually inverse isomorphisms ensuring that

$$L_K(E) \cong L_K(E) \oplus L_K(E).$$

The graph E is said to be the **rose** with two petals.

Thus, $L_K(E)$ is an example of a K-algebra R such that $R \oplus R \cong R$. In fact, more is true: $L_K(E)$ is a *universal* example of such an algebra: any such algebra R contains elements x, y and x', y' which satisfy the analogues of CK1 and CK2. This enables us to define a unital $*$-homomorphism $L_K(E) \to R$ that maps e and f to x and y (thus e^* to x' and f^* to y').

This example generalizes to the **rose** with n petals  . If E is this graph, $L_K(E)$ has the property that $L_K(E)^n \cong L_K(E)$ and it is the universal K-algebra with this property (in the same sense as when $n = 2$).

Besides the roses, there are other graphs whose Leavitt path algebras are not sums of matricial algebras over commutative rings. We mention some of them.

Example 12. If E is the graph $\bullet^v \xleftarrow{e} \bullet^w \circlearrowright f$, then $L_K(E)$ is isomorphic to the Toeplitz algebra $K[x,y|yx=1]$, the free K-algebra with generators x,y subject to the relation $yx=1$. Indeed, the map $\begin{array}{ll} e+f & \leftrightarrow x \\ e^*+f^* & \leftrightarrow y \end{array}$ extends to an isomorphism.

If E is the graph $\bullet^v \rightrightarrows \bullet^w$ with infinitely countably many edges $e_1, e_2, \ldots$ from v to w, then $L_K(E)$ is $*$-isomorphic to $\mathbb{M}_\infty(K) \oplus K1_\infty$ where $\mathbb{M}_\infty(K)$ stands for matrices with infinitely countably many rows and columns but with finitely many elements which are nonzero, and 1_∞ stands for the diagonal matrix with infinitely countably many rows and columns with 1 in all the diagonal entries. Indeed, the following map induces a $*$-isomorphism. $\begin{array}{lcl} w & \leftrightarrow & e_{11} \\ e_i & \leftrightarrow & e_{i+1\,1} \\ v & \leftrightarrow & 1_\infty - e_{11} \end{array}$

Note that in this representation $e_i e_i^* \leftrightarrow e_{i+1\,i+1}$.

Lecture 3. Graph monoids, graded rings and their Grothendieck groups

Next, we compute the Grothendieck group of a Leavitt path algebra. As we shall see, this group depends only on the graph and it is relatively straightforward to determine.

For simplicity, we will focus on graphs without infinite emitters, the **row-finite** graphs. For such graph E, define a monoid, called the **graph monoid** and denoted by M_E, as follows.

> M_E is a free abelian monoid on generators $[v]$ (think of $[v]$ as of the isomorphism class of the finitely generated projective module $vL_K(E)$) where v is a vertex subject to relations
>
> $$[v] = \sum_{e \in \mathbf{s}^{-1}(v)} [\mathbf{r}(e)] \qquad \text{if } v \text{ is regular.}$$

This means that the elements of M_E are finite sums of the generators and that the only relations that hold on these generators are those following from the relations on regular vertices listed in the definition above.

The monoid M_E turns out to be *isomorphic* to $\mathcal{V}(L_K(E))$ so, its group completion is isomorphic to $K_0(L_K(E))$. While we do not prove this fact here ([3] has more details), we can demonstrate that the defining relations of M_E holds on $\mathcal{V}(L_K(E))$. Let v be a regular vertex of E. Then,

$$\begin{array}{rll} [v] & = \left[\sum_{e\in\mathbf{s}^{-1}(v)} ee^*\right] & \text{(by CK2)} \\ & = \sum_{e\in\mathbf{s}^{-1}(v)} [ee^*] & \text{(definition of operation + in the monoid)} \\ & = \sum_{e\in\mathbf{s}^{-1}(v)} [e^*e] & \text{(left multiplication by } e \text{ is an iso of } e^*eL_K(E) \text{ and } ee^*L_K(E)) \\ & = \sum_{e\in\mathbf{s}^{-1}(v)} [\mathbf{r}(e)] & \text{(by CK1).} \end{array}$$

Example 13. (1) If E is $\bullet$, then M_E has one generator and no relations. Thinking of 1 as this generator, we have the monoid elements $0, 1, 1+1, 1+1+1, \ldots$ and so $M_E \cong \{0,1,2,\ldots\} = \mathbb{Z}^+$. The group completion is $K_0(L_K(E)) \cong \mathbb{Z}$.

(2) If E is $\bullet^v \circlearrowright$, then M_E has one generator $[v]$ and one relation $[v] = [v]$. Since this relation introduces nothing new, $M_E \cong \{0, 1, 2, \ldots\} = \mathbb{Z}^+$. The group completion is $K_0(L_K(E)) \cong \mathbb{Z}$.

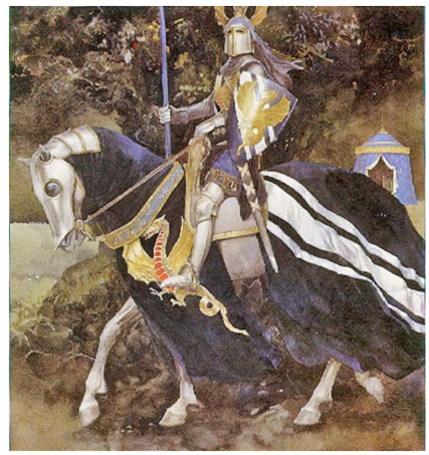
(3) If E is $\circlearrowleft_v \bullet \circlearrowright$, M_E has one generator $[v]$ and one relation $[v] = [v] + [v]$. When the cancellativity is imposed, this produces $0 = [v]$. Hence, $K_0(L_K(E)) = 0$.

The first two examples show that non-isomorphic Leavitt path algebras of very different graphs may have the same K_0-groups. The third example shows that

$$L_K(E) \not\cong 0 \qquad \text{but} \qquad K_0(L_K(E)) = 0.$$

All of the examples indicate that for Leavitt path algebras, $K_0(L_K(E)) \cong K_0(L_K(F))$ rarely implies $L_K(E) \cong L_K(F)$.

Grading comes to the rescue. Leavitt path algebra is also **graded**. Considering the grading can help us distinguish various algebras as well as their Grothendieck groups better.



Graded rings. A ring R is **graded** by a group Γ if the ring can be chopped up in pieces labeled by the elements of Γ such that the product of elements from the γ-th and the δ-th piece is in the $\gamma\delta$-th piece for any $\gamma, \delta \in \Gamma$.

R	$\ldots$	x	$\ldots$	y	$\ldots$	xy	$\ldots$
		R_γ		R_δ		$R_{\gamma\delta}$	

More formally, R is Γ-**graded** if $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ for additive subgroups R_γ and $R_\gamma R_\delta \subseteq R_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$.

ring

graded ring

If R, S are graded rings, a ring homomorphism $f : R \to S$ is a graded homomorphism if $f(R_\gamma) \subseteq S_\gamma$ for every $\gamma \in \Gamma$. We use $\cong_{\text{gr}}$ to denote that two rings are graded isomorphic.

Example 14. The ring of Laurent polynomials $K[x, x^{-1}]$ can be graded by $\mathbb{Z}$ so that the n-component consists of K-multiples of x^n.

$$K[x, x^{-1}]_n = Kx^n$$

We refer to this grading as the **natural grading** of $K[x, x^{-1}]$.

The same ring can also be graded **trivially** by $\mathbb{Z}$ so that

$$K[x, x^{-1}]_0 = K[x, x^{-1}] \text{ and } K[x, x^{-1}]_n = 0 \text{ for } n \neq 0.$$

Most importantly, a Leavitt path algebra can be graded by $\mathbb{Z}$ so that a path p of length n is in the n-component, and p^* is in $-n$ component. This induces the grading so that

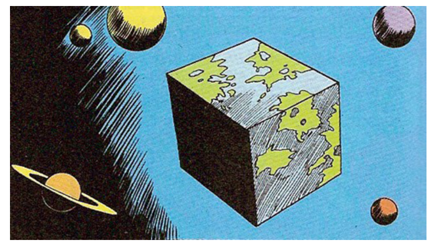
$$L_K(E)_n \text{ is the } K\text{-linear span of the set } \{pq^* \mid |p| - |q| = n\}$$

where $|p|$ denotes the *length* of the path p.

By the definition of a graded ring, every element is a finite sum of elements from components R_γ, for $\gamma \in \Gamma$. The elements of each component R_γ are said to be **homogeneous**.

One can generalize any ring-theoretic definition to graded rings by replacing "**element**" by "**homogeneous element**". For example,

field =	$\leftrightsquigarrow$	**graded field** =
every $x \neq 0$ has x^{-1}	$\leftrightsquigarrow$	every homog. $x \neq 0$ has x^{-1}
regular ring =	$\leftrightsquigarrow$	**graded regular ring** =
$(\forall x)\ x \in xRx$	$\leftrightsquigarrow$	$(\forall$ homog. $x)\ x \in xRx$
free module =	$\leftrightsquigarrow$	**graded free module** =
has basis	$\leftrightsquigarrow$	has homog. basis



This creates a level of additional flexibility. For example, the ring $K[x, x^{-1}]$ is not a field (e.g. $1+x$ does not have an inverse), but with the natural grading (the first example above), it is a *graded field* because every nonzero homogeneous element is of the form kx^n for some $n \in \mathbb{Z}$ and $0 \neq k \in K$ and $k^{-1}x^{-n}$ is the inverse of kx^n.

Graded modules and shifts. A left module M of a Γ-graded ring R is **graded** if $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ such that $R_\gamma M_\delta \subseteq M_{\gamma\delta}$. If M, N are two graded modules and $f : M \to N$ is a module homomorphism, then f is a graded homomorphism if $f(M_\gamma) \subseteq N_\gamma$ for every $\gamma \in \Gamma$. We use $\cong_{\mathrm{gr}}$ to denote an isomorphism of graded modules. For a graded module M and $\delta \in \Gamma$, the **δ-shift** $M(\delta)$ is the module M with the grading given by $M(\delta)_\gamma = M_{\gamma\delta}$.

A graded right module is analogously defined and the δ-shift of such a module M is written as $(\delta)M$ and its grading is given by $(\delta)M_\gamma = M_{\delta\gamma}$.

For example, let R be any ring trivially graded by $\mathbb{Z}$ (recall that this means that $R_0 = R$ and $R_n = 0$ for all $n \neq 0$). Considering R as a module over itself, its grading is as follows.

R	$\ldots$	0	0	R	0	0	0	$\ldots$
$\mathbb{Z}$	$\ldots$	-2	-1	0	1	2	3	$\ldots$

The module $R(1)$ is graded as follows.

$R(1)$	$\ldots$	0	R	0	0	0	0	$\ldots$
$\mathbb{Z}$	$\ldots$	-2	-1	0	1	2	3	$\ldots$

Finitely generated graded free and projective modules. Recall that a finitely generated free module is a finite sum of the form

$$R \oplus \ldots \oplus R.$$

We have seen that a *graded free* left module can be defined as a module which has a basis consisting of homogeneous elements. This implies that a **finitely generated graded free** left module has the form

$$R(\gamma_1) \oplus \ldots \oplus R(\gamma_n)$$

for some $\gamma_1, \ldots, \gamma_n \in \Gamma$. So, a **finitely generated graded projective** module is any direct summand of a module as above.

A finitely generated graded free *right* module has the form $(\gamma_1)R \oplus \ldots \oplus (\gamma_n)R$ for $\gamma_1, \ldots, \gamma_n \in \Gamma$.

Example 15. If Γ is the trivial group $\{1\}$, and K is any field, there is *just one* *one-dimensional free module*: K.

If $\Gamma = \mathbb{Z}$, for example, and K is Γ-graded, there can be *many one-dimensional graded free modules.*

$$\dots K(-3),\ K(-2),\ K(-1),\quad K(0),\quad K(1),\quad K(2),\quad K(3),\dots$$

Graded matrix rings. Recall that the matrix ring $\mathbb{M}_n(R)$ is isomorphic to the endomorphism ring of a finitely generated free right module.

$$\mathbb{M}_n(R) \;\cong\; \mathrm{End}_R\,(R^n)$$

The graded version of R^n depends on n-shifts of R which can help one understand the presence of n elements of Γ in the notation for the graded version of $\mathbb{M}_n(R)$

$$\mathbb{M}_n(R)(\gamma_1,\dots,\gamma_n).$$

The formula $\mathbb{M}_n(R) \;\cong\; \mathrm{End}_R\,(R^n)$ generalizes to

$$\mathbb{M}_n(R)(\gamma_1,\dots,\gamma_n) \quad \cong_{\mathrm{gr}} \quad \mathrm{End}_R\left((\gamma_1^{-1})R \oplus \dots \oplus (\gamma_n^{-1})R\right).$$

The endomorphism ring in the last formula is considered as a graded ring with its γ-component consisting of all endomorphisms f which map the δ-component into the $\gamma\delta$-component. The **graded matrix ring** $\mathbb{M}_n(R)(\gamma_1,\dots,\gamma_n)$ on the left side is defined as the (good old) ring of matrices $\mathbb{M}_n(R)$ with the grading given by

$$(r_{ij}) \in \mathbb{M}_n(R)(\gamma_1,\dots,\gamma_n)_\delta \text{ if and only if } r_{ij} \in R_{\gamma_i^{-1}\delta\gamma_j}.$$

Grading by the path lengths helps. Let us consider the following two graphs.

$$E = \quad \bullet \xrightarrow{e} \bullet \xrightarrow{f} \bullet_v \qquad\qquad F = \bullet \xrightarrow{e} \bullet_v \xleftarrow{f} \bullet$$

Since there are three paths ending at (one and only) sink of each graph, $L_K(E) \cong L_K(F) \cong \mathbb{M}_3(K)$. However, when grading is taken into consideration, *lengths of these paths matter.* For the first graph, the lenghts of paths v, f and ef are 0, 1, and 2, and for the second graph, the lengths of the paths v, e, f are 0, 1, and 1. Hence,

$$L_K(E) \cong_{\mathrm{gr}} \mathbb{M}_3(K)(0,1,2) \quad \not\cong_{\mathrm{gr}} \quad L_K(F) \cong_{\mathrm{gr}} \mathbb{M}_3(K)(0,1,1).$$

More generally, if E is a finite graph in which no cycle has an exit, its Leavitt path algebra is graded isomorphic to a direct sum of matrix algebras over graded fields K and $K[x^k, x^{-k}]$ for positive integers k. If E has n sinks $v_1,\dots,v_n$ and there are k_i paths p_{ij}, $j = 1,\dots,k_i$, $i = 1,\dots,n$, ending in v_i and m cycles $c_1,\dots,c_m$ and l_i paths q_{ij}, $j = 1,\dots,l_i$, $i = 1,\dots,m$, ending in a fixed arbitrary vertex of c_i, such that q_{ij} does not contain c_i. Then

$$L_K(E) \cong_{\mathrm{gr}} \bigoplus_{i=1}^{n} \mathbb{M}_{k_i}(K)(|p_{i1}|,\dots,|p_{ik_i}|) \oplus \bigoplus_{i=1}^{m} \mathbb{M}_{l_i}(K[x^{|c_i|}, x^{-|c_i|})(|q_{i1}|,\dots,|q_{il_i}|).$$

By [10, Theorem 1.3.3], the choice of a vertex in any of the cycles does not impact the graded isomorphism class of the algebra. Also, the lenghts in $\mathbb{M}_{k_i}(K)(|p_{i1}|,\dots,|p_{ik_i}|)$ can be permuted and the same holds for $\mathbb{M}_{l_i}(K[x^{|c_i|}, x^{-|c_i|})(|q_{i1}|,\dots,|q_{il_i}|)$.

Example 16. Let $E = \bullet^v \xrightarrow{e} \bullet^w \circlearrowright f$ and $F = \begin{tikzcd} \bullet^v \arrow[r, bend left, "e"] & \bullet^w \arrow[l, bend left, "f"] \end{tikzcd}$ which are the same two graphs as in Example 10. There are two paths, w and e, of E which end at w without counting the loop f. Hence, $L_K(E) \cong_{\mathrm{gr}} \mathbb{M}_2(K[x, x^{-1}])(0, 1)$. For F, if we consider the cycle ef to be based at v, then there are two paths, v and f, which end at v and do not contain entire ef. So, $L_K(F) \cong_{\mathrm{gr}} M_2(K[x^2, x^{-2}])(0, 1)$. One can show that $\mathbb{M}_2(K[x, x^{-1}])(0, 1) \not\cong_{\mathrm{gr}} \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$ (see [10, Proposition 1.4.4]), so we have that $L_K(E) \cong L_K(F) \cong \mathbb{M}_2(K[x, x^{-1}])$ but that

$$L_K(E) \cong_{\mathrm{gr}} M_2(K[x, x^{-1}])(0, 1) \quad \not\cong_{\mathrm{gr}} \quad L_K(F) \cong_{\mathrm{gr}} M_2(K[x^2, x^{-2}])(0, 1)$$

The Grothendieck group of a graded ring. If R is graded by Γ, $\mathcal{V}^\Gamma(R)$ denotes the set of *graded* isomorphism classes of finitely generated *graded* projective modules. Thus, $\mathcal{V}^\Gamma(R)$ is the "graded" version of $\mathcal{V}(R)$: the construction is exactly the same except that "projective" is replaced by "graded projective" (and one also shows that considering the left instead of the right modules produces isomorphic monoids). If right modules were considered when forming $\mathcal{V}^\Gamma(R)$, there is a Γ-action [1] compatible with $+$, given by $\gamma[P] = [(\gamma^{-1})P]$ for $\gamma \in \Gamma$. If left modules are considered, this action is $\quad \gamma[P] = [P(\gamma)]$.

The group completion of $\mathcal{V}^\Gamma(R)$ is denoted by $K_0^\Gamma(R)$ (sometimes $K_0^{\mathrm{gr}}(R)$) and we call it the **Grothendieck Γ-group.**

In the case when $\Gamma = \mathbb{Z}$, $K_0^\Gamma(R)$ is an abelian group with a $\mathbb{Z}$-action. Thus, there are *two actions of* $\mathbb{Z}$:

the *action of* $\mathbb{Z}$ producing the abelian group structure given by

$$m[R(n)] = [R(n)] + \ldots + [R(n)] = [R(n)^m]$$

and the *action of* $\mathbb{Z}$ due to the grading given by

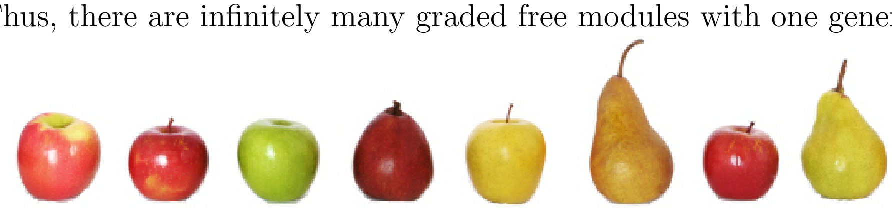
$$m[R(n)] = [R(n + m)].$$

Too many $\mathbb{Z}$s

To distinguish between these two actions, instead of $\Gamma = \mathbb{Z}$ with the additive notation, we use $\Gamma = \langle x \rangle = \{x^n \mid n \in \mathbb{Z}\}$ with the multiplicative notation where x is a new symbol. With this notation, $K_0^\Gamma(R)$ is a $\mathbb{Z}[x, x^{-1}]$-module where $\mathbb{Z}[x, x^{-1}]$ denotes the ring defined in the same way as $K[x, x^{-1}]$ except that the coefficients of its elements are elements of $\mathbb{Z}$, not K.

Let us compute K_0^Γ of two graded rings in the following example.

Example 17. Let $R = K[x, x^{-1}]$ and let us consider the trivial grading of R by $\mathbb{Z}$ (so that $R_0 = R$ and $R_n = 0$ for $n \neq 0$). Then $R(m) \not\cong_{\mathrm{gr}} R(n)$ for $n \neq m$ because $R(m)_{-m} = R$ and $R(n)_{-m} = 0$. Thus, there are infinitely many graded free modules with one generator



$$\ldots, R(-3), \quad R(-2), \quad R(-1), \quad R(0), \quad R(1), \quad R(2), \quad R(3), \quad R(4), \ldots$$

and the action of $\langle x \rangle$ on $[R(n)]$ is given by

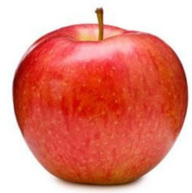
$$x^m[R(n)] = [R(n + m)].$$

[1]A group Γ acts on a set X if there is a map $f : G \times X \to X$ such that $f(1_\Gamma, x) = x$ and $f(\gamma, f(\delta, x)) = f(\gamma\delta, x)$.

Thus, the list $\ldots, [R(-3)],\ [R(-2)],\ [R(-1)],\ [R(0)],\ [R(1)],\ [R(2)],\ [R(3)], \ldots$ is equal to $\ldots, x^{-3}[R(0)],\ x^{-2}[R(0)],\ x^{-1}[R(0)],\ [R(0)],\ x[R(0)],\ x^2[R(0)],\ x^3[R(0)], \ldots$ which is in a bijective correspondence with the elements $\ldots, x^{-3},\ x^{-2},\ x^{-1},\ 1,\ x,\ x^2,\ x^3, \ldots$ of $\langle x\rangle$. Hence, every element of $\mathcal{V}^{\mathbb{Z}}(R)$ can be thought of as a finite linear combination of integer powers of x with nonnegative integers as coefficients. This monoid is denoted by $\mathbb{Z}^+[x, x^{-1}]$. Its group completion is $\mathbb{Z}[x, x^{-1}]$. Hence,

$$K_0^{\mathbb{Z}}(R) \cong \mathbb{Z}[x, x^{-1}].$$

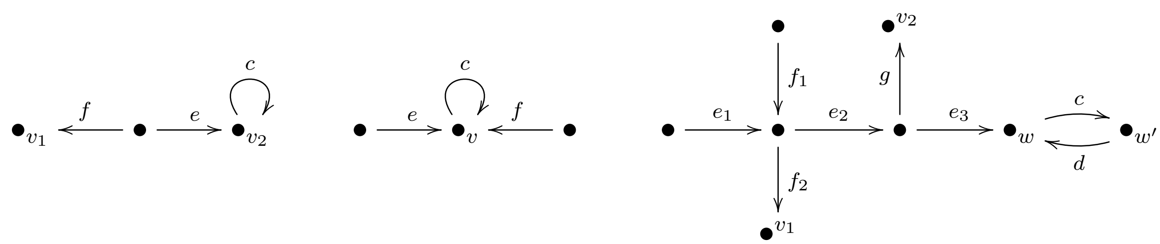
Example 18. Let $R = K[x, x^{-1}]$ again but let us grade R naturally (recall that this means that $K[x, x^{-1}]_n = K\{x^n\}$ for $n \in \mathbb{Z}$, see Example 14). Then



$R(m) \cong_{\text{gr}} R(n)$ for any m and n because right multiplication by x^{n-m} is a graded isomorphism $R(m) \cong_{\text{gr}} R(n)$. Thus, there is only one graded free module with one generator (up to a graded isomorphism). The generator x of the grade group $\langle x\rangle$ acts trivially on $R(0)$ since $x[R(0)] = [R(1)] = [R(0)]$ as $R(0) \cong_{\text{gr}} R(1)$. Thus, $\mathcal{V}^{\Gamma}(R)$ consists only of 0 and finite sums of $[R(0)]$. If we identify $[R(0)]$ with 1, we obtain the isomorphism $\mathcal{V}^{\mathbb{Z}}(R) \cong \mathbb{Z}^+$. Thus,

$$K_0^{\mathbb{Z}}(R) \cong \mathbb{Z}.$$

Exercise 10. Identify the Leavitt path algebras of the following graphs as *graded* matrix algebras over K and $K[x^n, x^{-n}]$ for some n or their direct sums.



Solutions to Exercise 10.

(1) The first graph has one sink and one cycle. There are two paths ending in v_1 and their lengths are 0 and 1. There are two paths ending in v_2 and their lengths are 0 and 1. Thus, $L_K(E) \cong_{\text{gr}} \mathbb{M}_2(K)(0,1) \oplus \mathbb{M}_2(K[x, x^{-1}])(0,1)$.

(2) The second graph has no sinks and a single cycle. There are three paths ending in v and their lengths are 0, 1, and 1. So, $L_K(E) \cong_{\text{gr}} \mathbb{M}_3(K[x, x^{-1}])(0,1,1)$.

(3) The third graph has two sinks v_1 and v_2 and one cycle cd. There are 4 paths of lengths 0, 1, 2, and 2 ending in v_1, and 5 paths of lengths 0, 1, 2, 3, and 3 ending in v_2. Choosing w for the base of cd, there are 6 paths of lengths 0, 1, 1, 2, 3, and 3. Thus,

$$L_K(E) \cong \mathbb{M}_4(K)(0,1,2,2) \oplus \mathbb{M}_5(K)(0,1,2,3,3) \oplus \mathbb{M}_6(K[x, x^{-1}])(0,1,1,2,3,3).$$

One eventually has to convince oneself that choosing w' instead of w one would get an algebra graded isomorphic to the one above. [10, Theorem 1.3.3] can help you understand why.

Lecture 4. The Grothendieck group of a graded graph algebra

For simplicity, we again consider row-finite graphs only and recall that all the concepts we consider can be extended to arbitrary graphs.

If E is a graph, recall that the graph monoid M_E is the free monoid on generators $[v]$ for $v \in E^0$ subject to the relations

$$[v] = \sum_{e \in \mathbf{s}^{-1}(v)} [\mathbf{r}(e)] \text{ for } v \text{ regular.}$$

We introduce the Γ-version. For the group $\Gamma = \langle x \rangle \cong \mathbb{Z}$ and a graph E, one introduces a monoid which some call **the graph Γ-monoid** because of its Γ-action while others call it **the talented monoid** because of the reasons we discuss after the definition. We denote this monoid by T_E.

The monoid T_E is a free abelian Γ-monoid on generators $[v]$ for $v \in E^0$ subject to the relations

$$[v] = \sum_{e \in \mathbf{s}^{-1}(v)} x\ [\mathbf{r}(e)] \text{ for } v \text{ regular.}$$

Note that the defining relation of M_E is modified by adding just one x in the formula from before. *Why x?* Because $x = x^1$ and 1 is the length of the path e from v to $\mathbf{r}(e)$. Thus, the "talent" of this monoid is that it *detects the number as well as the lengths of paths* between vertices of the graph.

Example 19. Let E be $\bullet_u \xrightarrow{e} \bullet_v \xrightarrow{f} \bullet_w$. The monoid T_E has three generators $[u], [v]$, and $[w]$ and two defining relations

$$[u] = x[v],\ [v] = x[w]$$

which imply that $[u] = x^2[w]$ indicating that u and w are connected by exactly *one path of length two.*

Let F be
$$\begin{tikzcd} \bullet_u \arrow[r] \arrow[d] & \bullet_v \arrow[dl] \\ \bullet_w & \end{tikzcd}$$
so that T_F has three generators and two defining relations, $[v] = x[w]$ and $[u] = x[v] + x[w]$. Thus, we also have that $\quad [u] = x[v] + x[w] = x^2[w] + x[w]$ indicating that u and w are connected by exactly *two paths, one of length 1 and the other of length 2.*

The monoid T_E is isomorphic to $\mathcal{V}^\Gamma(L_K(E))$ for any E. We denote its group completion by G_E and call it the **the graph Γ-group**. It is isomorphic to

$$K_0^\Gamma(L_K(E)).$$

Let us compute the talented monoid and its graph Γ-group of graphs we considered before in Examples 8, 9, and 11.

Example 20. (1) Let E be $\bullet$. Then T_E has one generator and no relations. Thus, $T_E \cong \mathbb{Z}^+\langle \dots x^{-2}, x^{-1}, 1, x, x^2, \dots \rangle = \mathbb{Z}^+[x, x^{-1}]$. Hence $G_E \cong \mathbb{Z}[x, x^{-1}]$.

(2) Let E be $\bullet^v \circlearrowright$. Then T_E has one generator $[v]$ and one relation $[v] = x[v]$ so that $\dots = x^{-1}[v] = [v] = x[v] = x^2[v] = \dots$ which implies that the action of $\langle x \rangle$ on T_E is trivial. If we think of $[v]$ as of 1, we have that $T_E \cong \mathbb{Z}^+$ and $G_E \cong \mathbb{Z}$.

Recall that the ordinary K_0-group does not distinguish between $L_K(\bullet)$ and $L_K(\bullet \circlearrowright)$.

(3) If E is $\bullet^{v}$, T_E has one generator $[v]$ and one defining relation $[v] = x[v] + x[v]$. Since $1 = \frac{1}{2} + \frac{1}{2}$, we can identify $[v]$ with 1 and the action of x with the multiplication by $\frac{1}{2}$. This creates an isomorphism between G_E and the group of dyadic rationals

$$\mathbb{Z}[\tfrac{1}{2}] = \{\tfrac{m}{2^n} \mid m \in \mathbb{Z}, n \in \mathbb{Z}^+\}$$

Recall that the ordinary K_0-group of $L_K(E)$ is zero.

By the first two examples, K_0^Γ distinguishes the graph algebras when K_0 does not. By the third example, K_0^Γ can be nontrivial when K_0 is trivial. This indicates that K_0^Γ is more telling.

Exercise 11. Determine the K_0^Γ-group of the graph $E = \bullet^v \longrightarrow \bullet^w$.

Solutions to Exercise 11. The monoid T_E has two generators $[v]$ and $[w]$ which are related by $[v] = x[w]$ and $[w] = x[w]$. The first relation enables you to reduce the number of generators to one and the second implies that the action of $\mathbb{Z}$ on that generator is trivial. Hence, $T_E \cong \mathbb{Z}^+$ and $K_0^\Gamma(L_K(E)) \cong G_E \cong \mathbb{Z}$.

As we shall see in a bit, consideration of order-units distinguishes the Γ-group of this graph from the Γ-group of the graph $\bullet$ which we also determined to be $\mathbb{Z}$.

The Graded Classification Conjecture. All seemed to be going so well with K_0^Γ of Leavitt path algebras that Roozbeh Hazrat formed the following question in [11].

Is it the case that for any two graphs E and F,

$$L_K(E) \cong_{\text{gr}} L_K(F)$$
as graded algebras
if and only if
$$K_0^\Gamma(L_K(E)) \cong K_0^\Gamma(L_K(F))$$
as pointed Γ-groups?

Before discussing the current state of the conjecture, let us look into "pointed". So, we have seen that $K_0^\Gamma(L_K(E))$ is an abelian group with a Γ-action. In addition, it is also a *pre-ordered* group.

The pre-order. A pre-order is any binary relation which is reflexive and transitive. Going back to an arbitrary ring and its $\mathcal{V}(R)$, note that there is a pre-order $\leq$ given by

$$[P] \leq [Q] \text{ if } P \text{ is isomorphic to a summand of } Q.$$

If R is a unital (or a locally unital) ring, the element $[R]$ of $\mathcal{V}(R)$ is said to be an **order-unit** because for every $[P] \in \mathcal{V}(R)$, there is $n \in \mathbb{Z}^+$ such that $[P] \leq n[R]$ (because every finitely generated projective module is a submodule of a finitely generated free module). When $K_0(R)$ is considered together with an order-unit, it is said to be **pointed**.

If $\Gamma = \langle x \rangle$, this concept generalizes as follows. The element $[R]$ of $\mathcal{V}^\Gamma(R)$ for a Γ-graded ring R is an **order-unit** because for every $[P] \in \mathcal{V}^\Gamma(R)$, there is an element $a \in \mathbb{Z}^+[x, x^{-1}]$ such that $[P] \leq a[R]$ (if P is a direct summand of $R(\gamma_1) \oplus \ldots \oplus R(\gamma_n)$, then think of a as $a = \sum_{i=1}^n \gamma_i$). $K_0^\Gamma(R)$ considered with an order-unit $[R]$ is also said to be **pointed**.

If E^0 is finite, the sum of all its vertices is the identity in the ring so the sum $\sum_{v\in E^0}[v]$ corresponds to the order-unit $[L_K(E)]$ under the isomorphism $T_E \cong \mathcal{V}^\Gamma(L_K(E))$.

The following example illustrates that "being pointed" matters for the Graded Classification Conjecture.

Example 21. Let $E = \bullet_u \longrightarrow \bullet_v \longrightarrow \bullet_w$ and $F = \bullet_u \longrightarrow \bullet_v \longleftarrow \bullet_w$. Then, T_E has three generators related by $[u] = x[v] = x^2[w]$, so we can take $[w]$ to be the only generator having no relations. Thus, $G_E \cong \mathbb{Z}[x, x^{-1}]$. T_F has three generators related by $[u] = x[v]$ and $[w] = x[v]$, so we can take $[v]$ to be the only generator having no relations. Thus, $G_F \cong \mathbb{Z}[x, x^{-1}]$ also. So, without considering the order-units, the two groups are indistinguishable. However, we consider G_E with the order-unit $[u] + [v] + [w] = x^2[w] + x[w] + [w]$ which maps to $x^2 + x + 1$ under the isomorphism $G_E \cong \mathbb{Z}[x, x^{-1}]$. We consider G_F with the order-unit $[u] + [v] + [w] = x[v] + [v] + x[v] = (2x + 1)[v]$ which we identify with $2x + 1$ under the isomorphism $G_F \cong \mathbb{Z}[x, x^{-1}]$. Hence, we have that

$$(K_0^\Gamma(L_K(E)), [L_K(E)]) \cong (\mathbb{Z}[x, x^{-1}], x^2{+}x{+}1) \quad \ncong \quad (K_0^\Gamma(L_K(F)), [L_K(F)]) \cong (\mathbb{Z}[x, x^{-1}], 2x{+}1).$$

Exercise 12. Compute the pointed Γ-groups of the following two graphs.

$$E = \begin{tikzcd} \bullet_u \arrow[r] & \bullet_v \arrow[loop right] \end{tikzcd} \text{ and } F = \begin{tikzcd} \bullet_u \arrow[r, bend left] & \bullet_v \arrow[l, bend left] \end{tikzcd}.$$

Solution to Exercise 12. T_E has two generators related by $[u] = x[v]$ and $[v] = x[v]$, so we can take $[v]$ to be the only generator and have x act trivially on it. Thus, $G_E \cong \mathbb{Z}$ and we consider it with the order-unit $[u] + [v] = x[v] + [v] = [v] + [v] = 2[v]$. Hence,

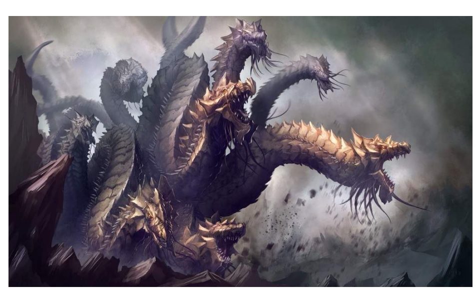
$$(K_0^\Gamma(L_K(E)), [L_K(E)]) \cong (\mathbb{Z}, 2).$$

T_F also has two generators and they are related by $[u] = x[v]$ and $[v] = x[u]$. Hence, we can take $[u]$ to be the only generator and have x act on it so that $[u] = x^2[u]$. Thus $G_F \cong \mathbb{Z}[x]/(x^2 = 1)$ and we consider it with the order-unit $[u] + [v] = [u] + x[u] = (1 + x)[u]$. Hence,

$$(K_0^\Gamma(L_K(F)), [L_K(F)]) \cong (\mathbb{Z}[x]/(x^2 = 1), 1 + x).$$

The current state of the Graded Classification Conjecture. The following list presents some positive results on the conjecture, listed in the chronological order.

Hazrat ([11]) – the conjecture holds for finite **polycephaly** graphs (every path leads to a sink, a rose, or a cycle with no exits). The polycephaly graphs are similar to no-exit graphs we considered before, except that they can have also roses instead of some of their sinks.



Ara and Pardo ([4]) – a weaker version of the conjecture holds for finite graphs without sources and sinks.

Hazrat and Vaš ([14]) – the involutive version of the conjecture holds for row-finite, no-exit graphs in which every infinite path ends in a sink or a cycle.

Eilers, Ruiz, Sims ([8]) – the conjecture and its C^*-algebra version hold for countable "amplified" graphs.

Relation with the Graded Isomorphism Conjecture. The **Isomorphism Conjecture** is asking whether

$$L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F) \text{ as rings } \text{ if and only if } \quad C^*(E) \cong C^*(F) \text{ as } *\text{-algebras}$$

and it has been formulated by Gene Abrams and Mark Tomforde in [2].

Mark Tomforde

Gene Abrams (in the middle)

Note that the implication

$$L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F) \text{ as } *\text{-algebras } \Rightarrow \; C^*(E) \cong C^*(F) \text{ as } *\text{-algebras}$$

always holds. Hence, to prove the conjecture, one would need the implication

$$L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F) \text{ as rings } \Rightarrow \; L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F) \text{ as } *\text{-algebras.}$$

Because of this, the term **Isomorphism Conjecture** sometimes is also used to denote the conjecture that, for any field K,

$$L_K(E) \cong L_K(F) \text{ as rings } \text{ if and only if } \quad L_K(E) \cong L_K(F) \text{ as } *\text{-algebras.}$$

The graded version of this statement, referred to as the **Graded Isomorphism Conjecture** stipulates that

$$L_K(E) \cong L_K(F) \text{ as } \textit{graded} \text{ rings}$$
$$\text{if and only if}$$
$$L_K(E) \cong L_K(F) \text{ as } \textit{graded} \text{ algebras.}$$

We have that

$$\textbf{Graded Classification Conjecture} \quad \Longrightarrow \quad \textbf{Graded Isomorphism Conjecture}.$$

Indeed, assuming that the Graded Classification Conjecture holds and that two Leavitt path algebras are graded isomorphic as graded rings, we have that their pointed K_0^Γ-groups are isomorphic. The Graded Classification Conjecture then implies that the two Leavitt path algebras are graded isomorphic as graded algebras.

Lecture 5. Irreducible representations of Leavitt path algebras

So far, we focused on classification of algebras based on their projective, the most "regular", modules. We switch our focus to the study of algebras based on their *"simplest"* modules now.

Recall that each representation $f : R \to \mathrm{End}_K(M)$ of a K-algebra R with M as its representation space determines the R-module structure on M by $rm = f(r)(m)$. Conversely, any left R-module M determines a representation of R $R \to \mathrm{End}_K(M)$ given by $r \mapsto (m \mapsto rm)$. This correspondence establishes a bijection between the representations of a K-algebra R and the left R-modules. Two representations f, g are *equivalent* if there is a K-space isomorphism ϕ of their representation spaces such that $\phi f(r) = g(r)\phi$ for every $r \in R$. This requirement is equivalent to the condition that the two representation spaces are isomorphic as R-modules.

In this correspondence, the *irreducible* representations correspond to the *simple* modules (modules without any nontrivial and proper submodules). Hence, depending on your taste, you can approach this last lecture as *a lecture on irreducible representations or a lecture on simple modules* of Leavitt path algebras.

If two representations are equivalent, their kernels are *equal*, equivalently, the annihilators of the representation spaces are equal. Recall than the *annihilator* of a left R-module M is the set of all $r \in R$ such that $rm = 0$ for all $m \in M$. One can show that this set is a two-sided ideal. In some cases, the equivalence of two algebra representations is used in the sense that the kernels of the two representations are equal.

Consideration of *branching systems*, initiated in [9], provides a consistent way to create large classes of left $L_K(E)$-modules. In particular, up to the equality of their annihilators, all graded simple modules can be introduced using branching systems as we shall soon see.

A **branching system** of a graph E is a set X with its subsets X_v for $v \in E^0$ and X_e for $e \in E^1$, and maps $\sigma_e : X_{\mathbf{r}(e)} \to X_e$ for $e \in E^1$ such that the following holds.

(1) The sets $X_v, v \in E^0$, are mutually disjoint and the sets $X_e, e \in E^1$, are mutually disjoint.
(2) $X_e \subseteq X_{\mathbf{s}(e)}$ for every $e \in E^1$.
(3) The map σ_e is a bijection for every $e \in E^1$.
(4) $X_v = \bigcup_{e \in \mathbf{s}^{-1}(v)} X_e$ for each regular vertex v.

If (4) holds also for infinite emitters, X is **perfect**. If $X = \bigcup_{v \in E^0} X_v$, X is **saturated**.

For a branching system X, one defines a left $L_K(E)$-module $M(X)$ as follows: $M(X)$ is the vector space over K with basis X and with the $L_K(E)$-module structure induced by

$$\begin{array}{llll} v \cdot x = x & \text{if } x \in X_v & \text{and} \quad v \cdot x = 0 & \text{otherwise,} \\ e \cdot x = \sigma_e(x) & \text{if } x \in X_{\mathbf{r}(e)} & \text{and} \quad e \cdot x = 0 & \text{otherwise, and} \\ e^* \cdot x = \sigma_e^{-1}(x) & \text{if } x \in X_e & \text{and} \quad e^* \cdot x = 0 & \text{otherwise} \end{array}$$

for $v \in E^0, e \in E^1$, and $x \in X$. Why is $M(X)$ a $L_K(E)$-module? Define a map f on $E^0 \cup E^1$ to $\mathrm{End}_K(M(X))$ by mapping $v \in E^0$ to $m \mapsto m \cdot v$, $e \in E^1$ to $m \mapsto e \cdot m$, and e^* to $m \mapsto e^* \cdot m$. One shows that the images $f(v)$, $f(e)$, and $f(e^*)$ satisfy V, E1, E2, CK1, and CK2, and the *Universal Property of* $L_K(E)$ then implies that f uniquely extends to a K-*algebra homomorphism* $L_K(E) \to \mathrm{End}_K(M(X))$. Hence, $M(X)$ is a $L_K(E)$-module.

By [13], a branching system is said to be **graded** if there is a map $\deg : X \to \mathbb{Z}$ such that $\deg(\sigma_e(x)) = \deg(x) + 1$. If X is graded, then $M(X)$ is graded where $M(X)_n$ for $n \in \mathbb{Z}$ is the K-linear span of elements $x \in X$ with $\deg(x) = n$.

We present some examples of branching systems. As we shall see, these are also examples of irreducible representations of a Leavitt path algebra. For examples of different flavor, see [9].

Example 22. (1) **The sink type.** Let v be a sink and X be the set of all paths in the path algebra $P_K(E)$ which end at v.

For $w \in E^0$ and $e \in E^1$, we let
$X_w = \{p \in X \mid \mathbf{s}(p) = w\}$,
$X_e = \{p \in X \mid p = eq$ for some $q \in X\}$,
$\sigma_e : X_{\mathbf{r}(e)} \to X_e$ be given by $p \mapsto ep$.
$\deg : X \to \mathbb{Z}$ be given by $p \mapsto |p|$.

As an exercise, show that X is a graded, perfect and saturated branching system. It is relatively direct to check that the first two axioms hold, that X is saturated and that the map given by $ep \mapsto p$ is the inverse of σ_e. To check the fourth axiom and to show that X is perfect, note that if $w \in E^0$ connects to v, then any path p that w emits to v has length larger than one and if e is the first edge of p, then $p \in X_e$. This indicates that $X_w \subseteq \bigcup_{e\in\mathbf{s}^{-1}(w)} X_e$. The converse is direct. If w does not connect to v, then X_w is the empty set and $\bigcup_{e\in\mathbf{s}^{-1}(w)} X_e$ is the empty set also. X is graded since $\deg(ep) = |ep| = 1 + |p| = 1 + \deg(p)$ for any $p \in X$.

(2) **The infinite-emitter type.** Let v be an infinite emitter and X, X_w, X_e, σ_e, and deg be defined as for the previous type. One checks that X is a graded and saturated branching system similarly as in the previous example. X is not perfect since $v \in X_v$, but $v \notin X_e$ for any $e \in E^1$ ($v \neq ep$ any $p \in X$ and $e \in E^1$).

(3) **The infinite-path type.** An *infinite path* is a sequence of edges $e_1e_2\ldots$ where each finite prefix is a path as defined before. Two infinite paths α and β are *tail equivalent* if $\alpha = p\gamma$ and $\beta = q\gamma$ for some paths p and q and an infinite path γ. For an infinite path α, let X be the set of all infinite paths tail equivalent to α.

For $v \in E^0$ and $e \in E^1$, we let
$X_v = \{\beta \in X \mid \mathbf{s}(\beta) = v\}$,
$X_e = \{\beta \in X \mid \beta = e\gamma$ for some $\gamma \in X\}$,
$\sigma_e : X_{\mathbf{r}(e)} \to X_e$ be given by $\beta \mapsto e\beta$.

One checks that X is a perfect and saturated branching system similarly as in the previous examples. By [13], $V_{[\alpha]}$ is graded if and only if α is *irrational*, i.e. not tail equivalent to $ccc\ldots$ for a cycle c. If α is such, $\deg : X \to \mathbb{Z}$ given by $\beta \mapsto |q| - |p|$ where p and q are the shortest prefixes of α and β such that $\alpha = p\gamma$ and $\beta = q\gamma$ for the same infinite path γ. If α is not irrational, one can show that no degree function can be defined since $\beta = c\beta$ for $\beta = ccc\ldots$.

(4) **The twisted-infinite-path type.** This type consists of modules denoted by $V^f_{[\alpha]}$ where α is an infinite rational path and f an irreducible polynomial over $K[x, x^{-1}]$. [5] contains more details. The modules of this type are not graded.

We say that a module is a **Chen module** if it is a module of one of the four types of the above example. The Chen modules of different types are not isomorphic. By [7], [5], and [18], Chen modules are *simple*. The proofs for different types rely on the fact that if N is a submodule of a Chen module M, and if N contains an element $a \neq 0$, then a $L_K(E)$-multiple of a produces the generating element of M : a sink, an infinite emitter, or an infinite path. While we will not go into the full proof of this, you can convince yourself that this indeed happens in the case when a is a basis element (that is, an element of X if $M = M(X)$). For example, if v is a sink or an infinite emitter defining the module $M = N_v$, and a is a path $p \in X$ (so that $\mathbf{r}(p) = v$), then $p^* \cdot p = v$ showing that $v \in N$ since $a = p \in N$. This implies that $N = M$. Similarly, if α is an infinite path defining the module $M = V_{[\alpha]}$ and a is an infinite path $\beta = q\gamma$ where $\alpha = p\gamma$ for some paths p, q and an infinite path γ, then $\alpha = pq^* \cdot \beta \in N$ which implies that $N = M$.

Moreover, by [5], *any simple $L_K(E)$-module has annihilator equal to one of the Chen modules.* Thus, up to the equality of their annihilators, Chen modules represent all simple modules.

Graded simple modules. If R is a graded ring, a graded R-module M is **graded simple** if it has no nontrivial and proper *graded* submodules. For a Leavitt path algebra, we would like to determine the graded version of Chen modules – a short-list of graded simple modules which represent all graded simple modules up to the equality of their annihilators. It turns out that one does not have to add a lot to those Chen modules which are graded – just one more type.

(4^{gr}) **The exclusive-cycle type.** A cycle c of E is **exclusive** if "no exit returns" or, more formally, no vertex of c is the base of a cycle distinct from c. If c is an exclusive cycle and v is an arbitrary vertex of c, we define a branching system X as follows.

Let Y be the set of the basis elements of the path algebra $P_K(\widehat{E})$ of the extended graph $\widehat{E}$ which have the form pq^* where p and q are paths with $\mathbf{r}(p) = \mathbf{r}(q)$ in c and $\mathbf{s}(q) = v$. Note that such path q is completely in c because c is exclusive. We say that $pq^* \in Y$ is *not reduced* if p and q have positive lengths and if they end with the same edge e. This defines a function on Y, which we denote by red and call the *reduction function*, by

$$\mathrm{red}(ee^*q^*) = q^* \text{ and } \mathrm{red}(pee^*q^*) = \mathrm{red}(pq^*)$$

for an edge e in c and $ee^*q^*, pee^*q^* \in Y$. We say that $pq^* \in Y$ is *reduced* if $\mathrm{red}(pq^*) = pq^*$.

Let $X = \{pq^* \in Y \mid \mathrm{red}(pq^*) = pq^*\}$ be the set of elements of Y which are reduced. For $w \in E^0$ and $e \in E^1$, let

$X_w = \{pq^* \in X \mid \mathbf{s}(p) = w\}$,

$X_e = \{pq^* \in X \mid pq^* = \mathrm{red}(ers^*) \text{ for some } ers^* \in Y\}$,

$\sigma_e : X_{\mathbf{r}(e)} \to X_e$ be given by $pq^* \mapsto \text{red}(epq^*)$, and
$\deg : X \to \mathbb{Z}$ be given by $pq^* \mapsto |p| - |q|$.

This defines a graded, saturated and perfect branching system. Try to convince yourself that the first two axioms hold for X and that X is saturated. It may be helpful to note that $X_e = X_{\mathbf{s}(e)}$ for every edge e of cycle c. This also implies that every vertex of c satisfies the fourth axiom. For vertices which are not in c the argument that the fourth axiom holds and that X is perfect is more similar to that in the previous examples. Try also to show that the inverse of σ_e is the map given by $\text{red}(epq^*) \mapsto \text{red}(pq^*)$.

We use N_c^v to denote $M(X)$. If it is clear which vertex of c we selected, we use N_c.

The following exercise may help you get better acquainted with the reduction function.

Exercise 13. If $pq^* \in Y$ and $\mathbf{s}(p)$ is in c, then $p^* \cdot \text{red}(pq^*) = q^*$.

Solution to Exercise 13. We use induction on the length $|p|$. If p is a vertex w in c, then $w = \mathbf{r}(q)$ and $p^* \cdot \text{red}(pq^*) = w \cdot \text{red}(q^*) = w \cdot q^* = q^*$. Assuming the induction hypothesis for $pq^* \in Y$, let us show the claim for $epq^* \in Y$ where $e \in E^1$ is with $\mathbf{s}(e)$ in c. In this case,

$$(ep)^* \cdot \text{red}(epq^*) = p^* \cdot (e^* \cdot \text{red}(epq^*)) = p^* \cdot \sigma_e^{-1}(\text{red}(epq^*)) = p^* \cdot \text{red}(pq^*) = q^*.$$

If c has more than one vertex and w is another vertex of c, then $N_c^v \cong_{\text{gr}} N_c^w(n)$ where n is the length of the (unique) path from v to w in c. Thus, the choice of the vertex in c impacts the graded isomorphism type just up to a shift.

The argument that the module N_c is *graded simple* resembles the arguments used for the previous types. Assuming that M is a nontrivial submodule of N_c, we show that $M = N_c$ by showing that the existence of a nonzero element a in M implies that v is in M. We show this claim only if $a = \text{red}(pq^*)$ is an element of X (see [19] for full details). If $pq^* \in Y$ is such that $\mathbf{s}(p)$ is not in c, write p in the form tr where r is in c and no vertex of t except the last one is in c. Then $0 \neq \text{red}(rq^*) = t^* \cdot \text{red}(pq^*) \in M$, so we can reduce our consideration to $pq^* \in Y$ such that $\mathbf{s}(p)$ is in c. In this case, $qp^* \cdot \text{red}(pq^*) = q \cdot (p^* \cdot \text{red}(pq^*)) = q \cdot q^*$ by Exercise 13. So, we have that $v = \mathbf{s}(q) = \text{red}(qq^*) = q \cdot q^* = qp^* \cdot \text{red}(pq^*) \in M$ which implies that $M = N_c$.

So, N_c is graded simple. We note that N_c is *not simple.* While the full proof of this last claim is much more involved, we can present a portion of the argument. Namely, the existence of a cycle without exits is an obstruction for the simplicity. If F is the graph obtained from E considering only the paths ending at c, then c is without exits in F. For simplicity, assume that E is finite, so F is also finite. In this case, $L_K(F)$ is isomorphic to a matrix algebra over $K[x, x^{-1}]$. While graded simple, this ring, considered as a left module over itself, is not simple because the left ideal generated by $1 + x$ is neither trivial nor improper.

So, the modules N_v for v a sink or an infinite emitter, $V_{[\alpha]}$ for an irrational infinite path α, and N_c for an exclusive cycle c are *graded simple*. We refer to them as **graded Chen modules**. One checks that graded Chen modules of different types have different annihilators so that they are not isomorphic to each other.

The reference to "Chen modules" in the name "graded Chen modules" is justified by the fact that *each graded simple module has annihilator equal to the annihilator of a graded Chen* *module.* With just a bit more background, we could prove this fact, but since we are probably out of time at this point, we refer to [19] for full details.

A note on the references. I hope that these lectures inspired some further interest in the topics we covered. Below are some recommendations for possible future reading.

For more on the theory of rings and modules, see [16] and [17], on $*$-rings, see [6], and on graded rings, see [10]. For more on Leavitt path algebras, see [1].

For more on the graph monoid, see [3] and more on its graded version [12] and [15].

For more on the Graded Classification Conjecture, see [11], [4] [14], and [8] and on the Isomorphism Conjecture see [2].

For more on branching systems, see [9] and for more on irreducible representations of Leavitt path algebras see [7], [5], [18], [13], and [19].

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