Chaos on ultragraph shift spaces

Daniel Gonçalves - UFSC

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Coding dynamical systems

Edge shift space

Definition 1.

Given a graph (E^0, E^1, r, s) the associated edge shift space is defined as

$$X = \{(e_n) : r(e_n) = s(e_{n+1}) \forall n = 1, 2, \ldots\},$$

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Definition 2.

Metric on X:

$$d\left((e_n), (f_n)\right) = \frac{1}{2^j}, \text{ where } e_1 \dots e_{j-1} = f_1 \dots f_{j-1} \text{ and } e_j \neq f_j.$$

Let (X, f) be a dynamical system.

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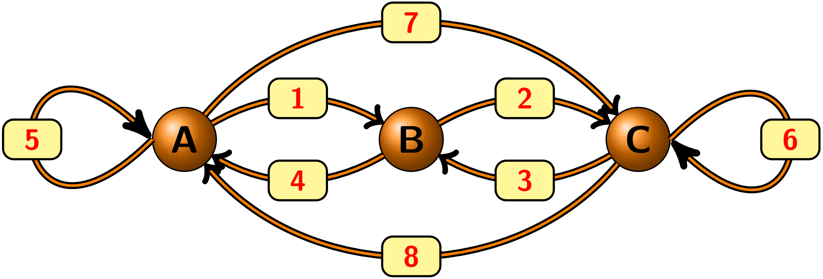
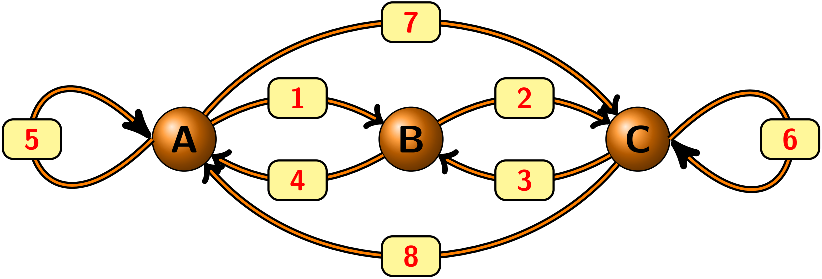
Example 4.

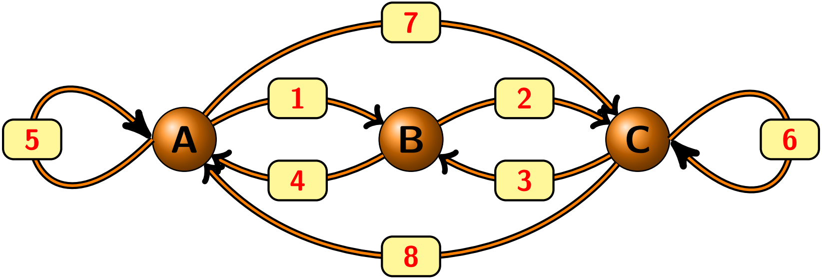
Full shift on two generators.

Shift spaces





How to define an analogue for infinite alphabets?



How to define an analogue for infinite alphabets?
Appeared most commonly in the context of countable-state Markov chains (or equivalently, shifts coming from countable directed graphs or matrices). Examples of such approach can be found in [8, 10, 18], [1, 2], etc..

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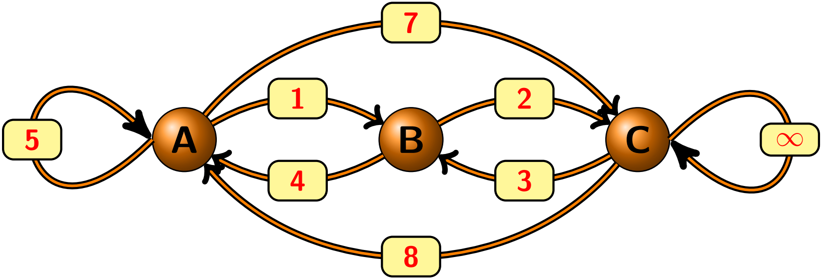
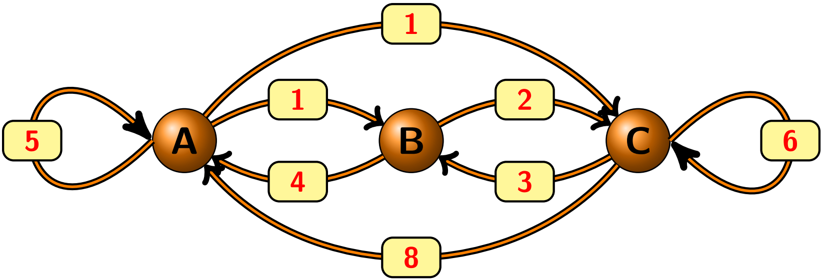
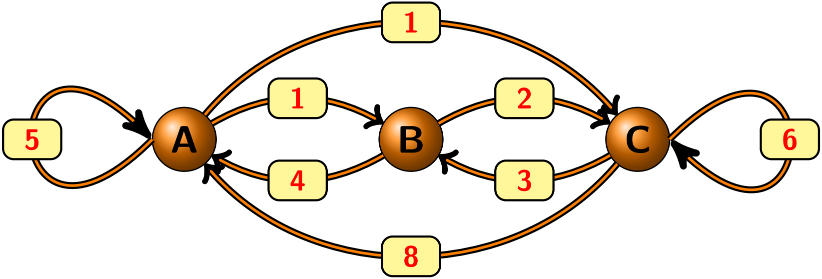
• We want a definition that puts the approaches connected to graph C*-algebras and topological Markov chains with infinitely many states under one umbrella.

Infinite graphs



Ultragraphs





Definition 2.1.

An ultragraph is a quadruple $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consisting of two countable sets $G^0, \mathcal{G}^1$, a map $s : \mathcal{G}^1 \to G^0$, and a map $r : \mathcal{G}^1 \to P(G^0) \setminus \{\emptyset\}$, where $P(G^0)$ stands for the power set of G^0.

Definition 2.2.

Let $\mathcal{G}$ be an ultragraph. Define $\mathcal{G}^0$ to be the smallest subset of $P(G^0)$ that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite unions and non-empty finite intersections.

Ultragraph C*-algebra

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Definition 2.3.

The ultragraph algebra $C^*(\mathcal{G})$ is the universal C^*-algebra generated by a family of partial isometries with orthogonal ranges $\{s_e : e \in \mathcal{G}^1\}$ and a family of projections $\{p_A : A \in \mathcal{G}^0\}$ satisfying

1. $p_\emptyset = 0, p_A p_B = p_{A\cap B}, p_{A\cup B} = p_A + p_B - p_{A\cap B}$, *for all* $A, B \in \mathcal{G}^0$;
2. $s_e^* s_e = p_{r(e)}$, *for all* $e \in \mathcal{G}^1$;
3. $s_e s_e^* \leq p_{s(e)}$ *for all* $e \in \mathcal{G}^1$; *and*
4. $p_v = \sum_{s(e)=v} s_e s_e^*$ *whenever* $0 < |s^{-1}(v)| < \infty$.

Example: Renewal shift

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & & \ddots & \end{pmatrix}$$

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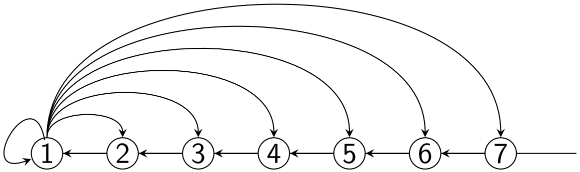

Definition 3.1.

For each subset A of G^0, let $\varepsilon(A)$ be the set $\{e \in \mathcal{G}^1 : s(e) \in A\}$. We shall say that a set A in $\mathcal{G}^0$ is an infinite emitter whenever $\varepsilon(A)$ is infinite.

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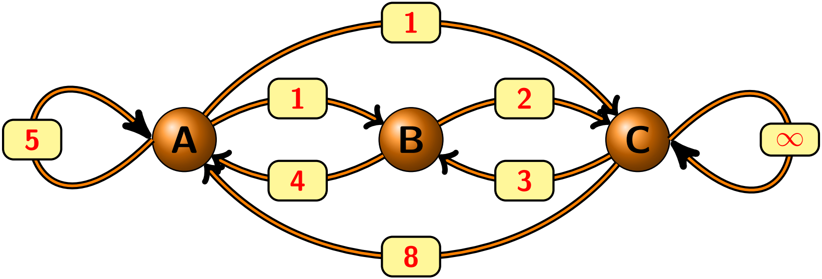
We denote the set of all minimal infinite emitters in $r(\alpha)$ by M_α.

The topological space



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Let

$$X_{fin} = \{(\alpha, A) \in \mathfrak{p} : |\alpha| \geq 1 \text{ and } A \in M_\alpha\} \cup$$

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Define

$$X = \mathfrak{p}^\infty \cup X_{fin}.$$

The topological space

Metric in X:

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list the elements of $\mathfrak{p}$ as $\mathfrak{p} = \{p_1, p_2, p_3, \ldots\}$. Then, for $x, y \in X$, we have that

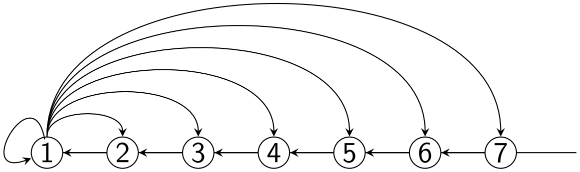
$$d_X(x,y) := \begin{cases} 1/2^i & i \in \mathbb{N} \text{ is the smallest value such that } p_i \text{ is an initial} \\ & \qquad \text{segment of one of } x \text{ or } y \text{ but not the other,} \\ 0 & \text{if } x = y. \end{cases} \tag{1}$$

Definition 3.3.

The shift map is the function $\sigma : X \to X$ defined by

$$\sigma(x) = \begin{cases} \gamma_2\gamma_3\dots & \textit{if } x = \gamma_1\gamma_2\dots \in \mathfrak{p}^\infty \\ (\gamma_2\dots\gamma_n, A) & \textit{if } x = (\gamma_1\dots\gamma_n, A) \in X_{fin} \textit{ and } |x| > 1 \\ (A, A) & \textit{if } x = (\gamma_1, A) \in X_{fin} \\ (A, A) & \textit{if } x = (A, A) \in X_{fin} \textit{ and } |x| = 0. \end{cases}$$

Back to the Renewal shift



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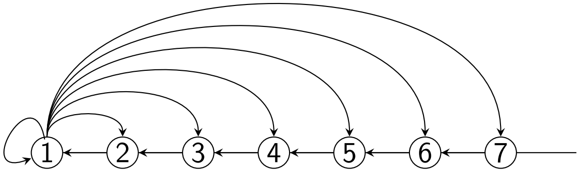

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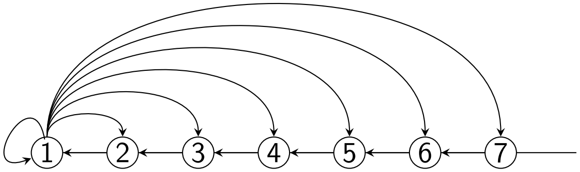

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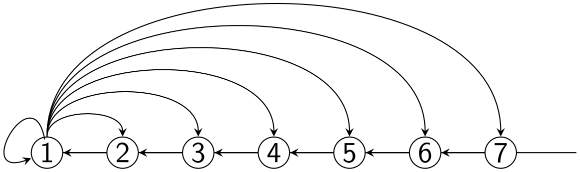

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• Only one infinite emitter: $A = r(e_1) = \{1, 2, 3, \dots\}$.
• X consists of infinite paths and finite paths of the form

$$(f_1 \dots f_k e_1, A),$$

that "end" at A.

Via partial actions we obtain...

Theorem 4.1.

Let $\mathcal{G}_1, \mathcal{G}_2$ be two ultragraphs such that their shift spaces, X and Y respectively, are conjugate via a conjugacy $\phi : X \to Y$ that preserves length.

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- $h_{ba^{-1}} : X_{ab^{-1}} \rightarrow X_{ba^{-1}}$

Skew ring and partial crossed product

Action h on (topological) space induces an action α :

$$\alpha_{ba^{-1}} : C(X_{ab^{-1}}) \to C(X_{ba^{-1}}), \text{ such that}$$

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$$C(X) * \mathbb{F} = \bigoplus_{g \in \mathbb{F}} C(X_g) = \{\sum_{\text{finite}} f_t \delta_t : f_t \in C(X_t)\}$$

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$L_K(E) \sim L_c(X) * \mathbb{F}$ *and* $C^*(E) \sim C(X) \ltimes \mathbb{F}$

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$L_c(X) * \mathbb{F}$ *is simple iff action is topologically free and minimal iff graph satisfies Condition (L) and there are no hereditary and saturated subsets of the vertices.*

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Theorem 8.

Let E_1, E_2 *be graphs. The following are equivalent:*

- X_{E_1} *and* X_{E_1} *are orbit equivalent.*
- *There is an isomorphism between* $C(X_{E_1}) \ltimes \mathbb{F}_1$ *and* $C(X_{E_2}) \ltimes \mathbb{F}_2$ *that takes* $C(X_{E_1})$ *to* $C(X_{E_2})$.
- *The associated partial actions are continuous orbit equivalent.*

Further topics

- Path groupoid

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