A Talented Monoid view on Lie Bracket algebra arising from Leavitt Path Algebra

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- ☺ A.N. Sebandal, J.P. Vilela, ***The Jordan-Hölder Theorem for monoids with group action***, Journal of Algebra and its Applications. doi.org/10.1142/S0219498823500883. (December 2021).

- ☺ R. Hazrat, A.N. Sebandal, J.P. Vilela, ***Graphs with disjoint cycles classification via the talented monoid***, Journal of Algebra. 593. 319-340. (November 2021).

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Graph

Talented monoid

Leavitt path algebra

$$\begin{tikzcd}
 & \text{Graph} \arrow[dl] \arrow[dr] & \\
\text{Talented monoid} & & \text{Leavitt Path algebra} \\
 & & \text{Lie Bracket Algebra} \arrow[u] \arrow[ull]
\end{tikzcd}$$

Definition

Let M be a monoid and Γ a group. M is said to be a Γ-*monoid* if there is an action of Γ on M.

Example

$\Gamma = \mathbb{Z}$ and $T = M_2(\mathbb{R})$ under pointwise addition

$$\phi : \mathbb{Z} \times T \longrightarrow T$$

$$\left(x, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \longmapsto \begin{pmatrix} e^x a & b \\ c & e^x d \end{pmatrix}$$

$\Rightarrow$ T is a $\mathbb{Z}$-monoid

Definition

A *Γ-order ideal* of a Γ-monoid M is a subset I of M such that for $\alpha, \beta \in \Gamma$,

$$^{\alpha}a + {}^{\beta}b \in I \iff a, b \in I.$$

Theorem

Let I and J be Γ-order-ideals of a commutative monoid T with $I \subseteq J$. Then

$$\left({}^{T}\big/_{I}\right)\Big/\left({}^{J}\big/_{I}\right) \cong {}^{T}\big/_{J}.$$

Theorem

Let I and J be Γ-order-ideals of a commutative monoid T with $I \subseteq J$. Then

$$\left(T \big/ I\right) \Big/ \left(J \big/ I\right) \cong T \big/ J.$$

Theorem

Let T be a refinement Γ-monoid and let Q, L and N be Γ-order-ideals of T such that $L \subseteq Q$. Then

$$Q \Big/ (L + (Q \cap N)) \cong (Q + N) \Big/ (L + N).$$

Definition

A commutative monoid T is called *refinement*, if for $a + b = c + d$, there exist $e_1, e_2, e_3, e_4 \in T$ such that

$$a = e_1 + e_2, \; b = e_3 + e_4 \text{ and } c = e_1 + e_3, \; d = e_2 + e_4.$$

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[Sebandal, Vilela (2021)]

$T = \{0, 1, x, y, z, s, b\}$ and an operation $(+)$ on given by

$+$	0	1	x	y	z	s	b
0	0	1	x	y	z	s	b
1	1	1	1	s	s	s	b
x	x	1	1	s	s	s	b
y	y	s	s	y	y	s	b
z	z	s	s	y	y	s	b
s	s	s	s	s	s	s	b
b	b	b	b	b	b	b	s

$+$	0	1	x	y	z	s	b
0	0	1	x	y	z	s	b
1	1	1	1	s	s	s	b
x	x	1	1	s	s	s	b
y	y	s	s	y	y	s	b
z	z	s	s	y	y	s	b
s	s	s	s	s	s	s	b
b	b	b	b	b	b	b	s

$$\Gamma = \{0\} \quad \Longrightarrow \quad T \text{ is a } \Gamma\text{-monoid}$$

+	0	1	x	y	z	s	b
0	0	1	x	y	z	s	b
1	1	1	1	s	s	s	b
x	x	1	1	s	s	s	b
y	y	s	s	y	y	s	b
z	z	s	s	y	y	s	b
s	s	s	s	s	s	s	b
b	b	b	b	b	b	b	s

$$\begin{aligned} \Gamma = \{0\} \quad &\Longrightarrow \quad T \text{ is a } \Gamma\text{-monoid} \\ 1+1 = x+x \text{ can not be refined} \quad &\Longrightarrow \quad T \text{ is not a refinement monoid} \\ A = \{0,1,x\},\ \ B = \{0,y,z\} \quad &\Longrightarrow \quad A \text{ and } B \text{ are} \\ &\qquad\ \Gamma\text{-order-ideals of } T \end{aligned}$$

$+$	0	1	x	y	z	s	b
0	0	1	x	y	z	s	b
1	1	1	1	s	s	s	b
x	x	1	1	s	s	s	b
y	y	s	s	y	y	s	b
z	z	s	s	y	y	s	b
s	s	s	s	s	s	s	b
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& \quad \Gamma\text{-order-ideals of } T
\end{aligned}$$

$$\begin{aligned}
b + b = s = x + y \in A + B \\
\text{but } b \notin A + B
\end{aligned}$$

+	0	1	x	y	z	s	b
0	0	1	x	y	z	s	b
1	1	1	1	s	s	s	b
x	x	1	1	s	s	s	b
y	y	s	s	y	y	s	b
z	z	s	s	y	y	s	b
s	s	s	s	s	s	s	b
b	b	b	b	b	b	b	s

$\Gamma = \{0\} \implies T$ is a Γ-monoid

$1 + 1 = x + x$ can not be refined $\implies T$ is not a refinement monoid

$A = \{0, 1, x\}, \ B = \{0, y, z\} \implies A$ and B are Γ-order-ideals of T

$b + b = s = x + y \in A + B$

but $b \notin A + B$

+	0	1	x	y	z	s	b
0	0	1	x	y	z	s	b
1	1	1	1	s	s	s	b
x	x	1	1	s	s	s	b
y	y	s	s	y	y	s	b
z	z	s	s	y	y	s	b
s	s	s	s	s	s	s	b
b	b	b	b	b	b	b	s

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A = \{0, 1, x\}, \;\; B = \{0, y, z\} \quad &\Longrightarrow \quad A \text{ and } B \text{ are} \\
& \quad \Gamma\text{-order-ideals of } T \\
b + b = s = x + y \in A + B \quad &\Longrightarrow \quad A + B \text{ is not a } \Gamma\text{-order-ideal} \\
\text{but } b \notin A + B \quad &
\end{aligned}$$

Theorem

Let T be a **refinement** Γ-monoid and let Q, L and N be Γ-order-ideals of T such that $L \subseteq Q$. Then

$$Q \Big/ (L + (Q \cap N)) \cong (Q + N) \Big/ (L + N).$$

Definition

Let I be a Γ-order-ideal of T. We say

(i) I is a *cyclic ideal* if for any $x \in I$, there is an $\alpha \in \Gamma$ such that ${}^{\alpha}x = x$;

(ii) I is a *comparable ideal* if for any $x \in I$, there is an $\alpha \in \Gamma$ such that ${}^{\alpha}x > x$;

(iii) I is a *non-comparable ideal* if for any $x \in I$, and any $\alpha \in \Gamma$, we have ${}^{\alpha}x \parallel x$.

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$x \geq y \iff x = y + a$ for some a

Example

Let $T = \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N}$ be a free abelian monoid with the action of $\mathbb{Z}$ on T defined by ${}^{1}(a, b, c, d) = (d, a, b, c)$ and extended to $\mathbb{Z}$.

$$x \in T \implies {}^{4}x = x$$

Then T is a cyclic $\mathbb{Z}$-monoid.

Let T be a Γ-order-ideal. A Γ-*series* for T is a sequence of Γ-order-ideals

$$0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = T. \qquad (*)$$

Furthermore, we say $(*)$ is a Γ-*composition series* if for each $i = 0, 1 \cdots, n-1$, $I_i \subsetneq I_{i+1}$ and each of quotients I_{i+1}/I_i are simple Γ-monoids.

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Γ- composition series | all simple quotients
cyclic | cyclic
comparable ⟷ comparable
noncomparable | noncomparable

We further say a composition series is of *mixed type* of certain kinds if the simple quotients are those given kinds.

Jordan-Hölder Theorem

Two Γ-series of a refinement Γ-monoid T have equivalent refinement. Thus, any Γ-composition series are equivalent and a Γ-monoid having a composition series determines a unique list of simple Γ-monoids.

Definition

A *directed graph* E is a tuple (E^0, E^1, r, s) where E^0 and E^1 are sets and r, s are maps from E^1 to E^0. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. We think of each $e \in E^1$ as an edge pointing from $s(e)$ to $r(e)$, that is, $s(e)$ is the *source* of e and $r(e)$ is the *range* of e. A graph E is *finite* if E^0 and E^1 are both finite.

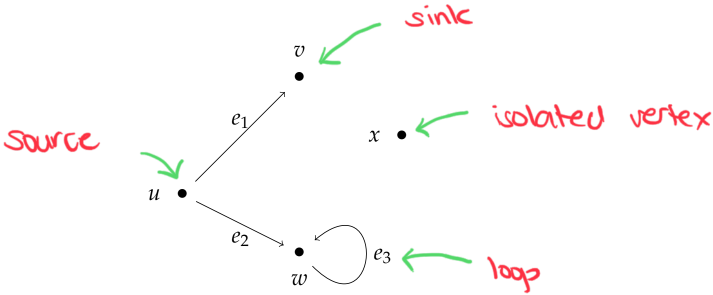

Figure : A graph E

Definition

A subset $H \subseteq E^0$ is said to be *hereditary* if for any $e \in E^1$, we have that $s(e) \in H$ implies $r(e) \in H$.

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$$u \circ\!\longrightarrow\!\circ\, v$$

$$u \in H \implies v \in H$$

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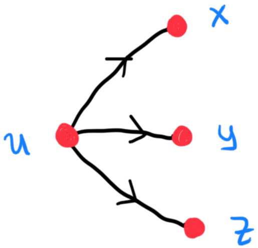
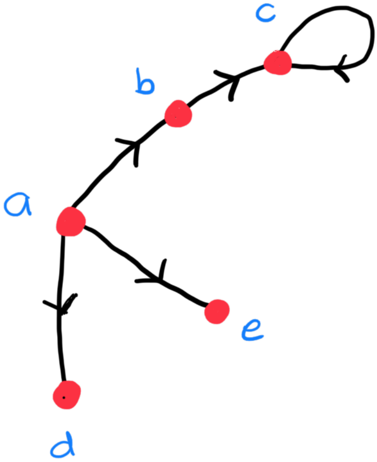
A subset $H \subseteq E^0$ is said to be *saturated* if for a regular vertex v, $r(s^{-1}(v)) \subseteq H$, then $v \in H$.



$x, y, z \in H \Rightarrow u \in H$

Example:



- $\{b, c\}$
- $\{d\}$
- $\{c\}$

Leavitt Path Algebra

For a row-finite graph E and a ring R with identity, we define the *Leavitt path algebra* of E, denoted by $L_R(E)$, to be the algebra generated by the sets $\{v : v \in E^0\}$, $\{\alpha : \alpha \in E^1\}$ and $\{\alpha^* : \alpha \in E^1\}$ with coefficients in R, subject to the relations

(V) $v_i v_j = \delta_{i,j} v_i$ for every $v_i, v_j \in E^0$;

(E) $s(e)e = e = er(e)$ and $r(e)e^* = e^* = e^* s(e)$ for all $e \in E^1$;

(CK1) $e^* e' = \delta_{e,e'} r(ea)$ for all $e, e' \in E^1$;

(CK2)

$$\sum_{\{\alpha \in E^1 : s(e) = v\}} ee^* = v$$

for every $v \in E^0$ which is not a sink.

Gelfand-Kirillov Dimension

Let A be an algebra (not necessarily unital), which is generated by a finite dimensional subspace V. Let V^n denote the span of all products $v_1 v_2 \cdots v_n$, $v_i \in V$, $k \leq n$. Then $V = V^1 \subseteq V^2 \subseteq \cdots$,

$$A = \bigcup_{n \geq 1} V^n \quad \text{and} \quad g_{V(n)} = \dim V^n < \infty.$$

Given the functions $f, g : \mathbb{N} \to \mathbb{R}^+$, if there exists $c \in \mathbb{N}$ such that $f(n) \leq cg(cn)$ for all $n \in \mathbb{N}$ we call f *asymptotically bounded* by g. If f is asymptotically bounded by g and g is asymptotically bounded by f, the functions f and g are said to be *asymptotically equivalent* denoted by $f \sim g$. The equivalence class of f under $\sim$ is called the *growth* of f.

If W is another finite-dimensional subspace that generated A, then $g_{V(n)} \sim g_{W(n)}$.

If $g_{(V(n))}$ is polynomially bounded, then the *Gelfand-Kirillov dimension* of A is defined as

$$\mathrm{GKdim} A = \limsup_{n \to \infty} \frac{\ln g_{V(n)}}{\ln n}.$$

The GK-dimension does not depend on a choice of the generating space V as long as $\dim V < \infty$. If the growth of A is not polynomially bounded, then $\mathrm{GKdim} A = \infty$.

Graph Monoid

Let E be a row-finite directed graph. The *graph monoid* of E, denoted by M_E, is the abelian monoid generated by $\{v : v \in E^0\}$, subject to

$$v = \sum_{e \in s^{-1}(v)} r(e)$$

for every $v \in E^0$ that is not a sink.

Talented Monoid

Let E be a row-finite graph. The *talented monoid* of E, denoted by T_E, is the abelian monoid generated by $\{v(i) : v \in E^0, i \in \mathbb{Z}\}$, subject to

$$v(i) = \sum_{e \in s^{-1}(v)} r(e)(i+1)$$

for every $i \in \mathbb{Z}$ and every $v \in E^0$ that is not a sink.

The additive group $\mathbb{Z}$ of integers acts on T_E via monoid automorphisms by shifting indices:

$$ {}^{n}v(i) = v(i+n) $$

$\rightarrow$ extends to an action of $\mathbb{Z}$ on T_E.

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$\Rightarrow T_E$ is a $\mathbb{Z}$-monoid

$v \in E^0 \longrightarrow v := v(0) = {}^{0}v$

$\Longrightarrow v(n) = {}^{n}v$

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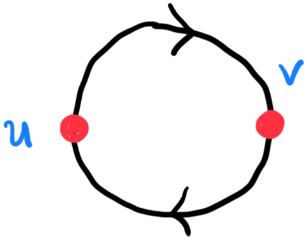
$\Rightarrow T_E$ is a refinement $\mathbb{Z}$-monoid

EXAMPLE:



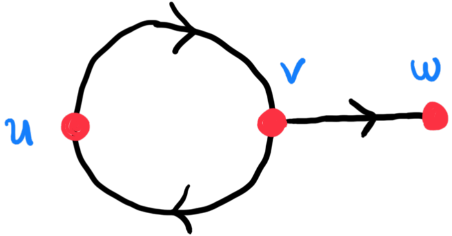
In M_E,

$$u = v = u + w$$

In T_E,

$$u(0) = v(1) = u(2)$$

$$\Rightarrow \quad u = {}^{2}u$$



$$u(0) = v(1) = w(2) + u(2)$$

$$\Longrightarrow \quad u >^{2} u$$

- Γ-monoid
- Leavitt Path algebra
- M_E
- T_E — refinement $\mathbb{Z}$-monoid

$$
\begin{tikzcd}
 & E \arrow[dl] \arrow[dr] & \\
L_K(E) \arrow[rr, dash, dashed] & & T_E
\end{tikzcd}
$$

- $\mathcal{L}(E)$ - hereditary saturated subsets of E
- $\mathcal{L}(T_E)$ - $\mathbb{Z}$-order-ideals of T_E
- $\mathcal{L}^{gr}(L_K(E))$ - graded ideal of $L_K(E)$

$$\mathcal{L}(T_E) \xleftarrow{\cong} \mathcal{L}(E) \xrightarrow{\cong} \mathcal{L}^{gr}(L_K(E))$$

$$\langle H \rangle \longleftarrow\!\shortmid\; H \;\longmapsto I(H)$$

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{Hereditary saturated sets}"] & \\ L_K(E) \arrow[rr, dash, dashed] & & T_E \end{tikzcd}$$

$$C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_n$$

- Chain of cycles

$$C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_n$$

- $\underline{d_1}$ - maximum length of a chain of cycles

- d_2 - maximum length of a chain of cycles with an exit

$$\Rightarrow \quad d_1 \geq d_2$$

Theorem [Alahmadi, Alsuhami, Jain, Zelmanov]

Let E be a finite graph.

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(2) If d_1 is the maximal length of a chain of cycles in E, and d_2 is the maximal length of chain of cycles with an exit, then

$$GKdim\ L_K(E) = \max(2d_1 - 1, 2d_2).$$

Theorem [Alahmadi, Alsuhami, Jain, Zelmanov]

Let E be a finite graph.

(1) $GKdim\ L_K(E) < \infty \iff E$ is a graph with disjoint cycles.

(2) If d_1 is the maximal length of a chain of cycles in E, and d_2 is the maximal length of chain of cycles with an exit, then

$$GKdim\ L_K(E) = \max(2d_1 - 1, 2d_2).$$

$$GKdim\ L_K(E) = \begin{cases} 2d_1 & \text{if } d_1 = d_2 \\ 2d_1 - 1 & \text{if } d_1 \neq d_2. \end{cases}$$

EXAMPLE:

$$
\begin{tikzcd}
E: & v \arrow[loop above] \arrow[r] \arrow[dr] & w \arrow[loop above] \\
 & & z
\end{tikzcd}
\qquad
\begin{tikzcd}
F: & v \arrow[loop above] \arrow[r] & w \arrow[loop above] \arrow[r] & z
\end{tikzcd}
$$

$d_1(E) = 2$

$d_2(E) = 1$

$\mathrm{GKdim}\, L_K(E) = 3$

$d_1(F) = 2$

$d_2(F) = 2$

$\mathrm{GKdim}\, L_K(F) = 4$

$$
\begin{tikzcd}
 & \text{chain of cycles} & \\
 & E \arrow[u] \arrow[dl] \arrow[dr] & \\
L_K(E) \arrow[rr, dash, dashed] \arrow[d] & & T_E \arrow[d] \\
\text{GK-dimension} & & \mathbb{Z}\text{- composition series}
\end{tikzcd}
$$

Theorem 12

Let E be a finite graph, $L_K(E)$ its associated Leavitt path algebra and T_E its talented monoid. Then the following are equivalent.

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(1) E is a graph with disjoint cycles;

(2) T_E has a composition series of cyclic and non-comparable types; monoid structure of T_E

Theorem 12

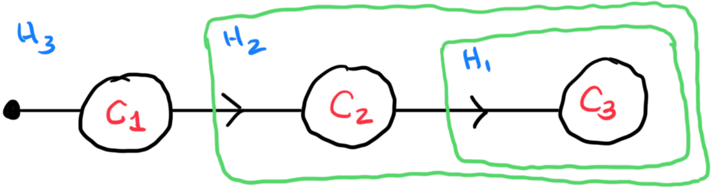
Let E be a finite graph, $L_K(E)$ its associated Leavitt path algebra and T_E its talented monoid. Then the following are equivalent.

(1) E is a graph with disjoint cycles;

(2) T_E has a composition series of cyclic and non-comparable types;

(3) $L_K(E)$ has a finite GK-dimension.

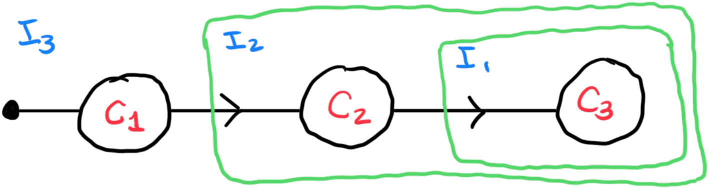
algebraic structure of $L_K(E)$

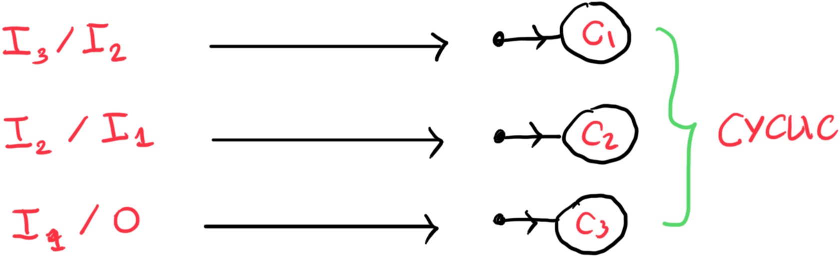


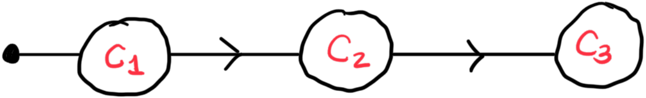
$$\emptyset \quad C \quad H_1 \quad C \quad H_2 \quad C \quad H_3$$

$$\downarrow$$

$$0 \quad C \quad I_1 \quad C \quad I_2 \quad C \quad I_3$$


I3
I2
I1
C1
C2
C3


I3 / I2
I2 / I1
I1 / O
C1
C2
C3
CYCLIC



$0 \quad C \quad I_1 \quad C \quad I_2 \quad C \quad I_3$

$d_1 = 3$

Consider the graphs with disjoint cycles below.

$$\begin{tikzcd} E: & v \arrow[loop above] \arrow[r] \arrow[dr] & w \arrow[loop above] \\ & & z \end{tikzcd} \qquad \begin{tikzcd} F: & v \arrow[loop above] \arrow[r] & w \arrow[loop above] \arrow[r] & z \end{tikzcd}$$

$GKdim\ L_K(E) = 3$ $\qquad$ $GKdim\ L_K(F) = 4$

Consider the graphs with disjoint cycles below.

$$E: \quad \begin{tikzcd} v \arrow[loop above] \arrow[rr] \arrow[dr] & & w \arrow[loop above] \\ & z & \end{tikzcd} \qquad\qquad F: \quad \begin{tikzcd} v \arrow[loop above] \arrow[r] & w \arrow[loop above] \arrow[r] & z \end{tikzcd}$$

$GKdim\ L_K(E) = 3$ $\qquad\qquad$ $GKdim\ L_K(F) = 4$

$$T_E \ncong T_F$$

- T_E has a periodic element $({}^1w = w)$
- T_F has no periodic elements (otherwise it has to have a cycle with no exit).

$$E: \quad \begin{tikzcd} v \arrow[loop above] \arrow[rr] \arrow[dr] & & w \arrow[loop above] \\ & z & \end{tikzcd} \qquad F: \quad \begin{tikzcd} v \arrow[loop above] \arrow[r] & w \arrow[loop above] \arrow[r] & z \end{tikzcd}$$

However, we have the following composition series for T_E and T_F, which are equivalent:

$$\langle z \rangle \subset \langle z, w \rangle \subset T_E$$
$$\langle z \rangle \subset \langle z, w \rangle \subset T_F.$$

Definition 10

Let T be a Γ-monoid. The *upper cyclic series* of T is a chain of Γ-order-ideals

$$0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n,$$

where I_{i+1}/I_i is the largest cyclic ideal of T/I_i, $0 \leq i \leq n-1$. We call I_n the *leading ideal* of the series and denote n by $l_c(T)$.

Theorem

Let E be a finite graph with disjoint cycles, $L_K(E)$ its associated Leavitt path algebra and T_E its talented monoid.

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Then

$$GKdim\ L_K(E) = \begin{cases} 2d_1 & \text{if } d_1 = d_2 \\ 2d_1 - 1 & \text{if } d_1 \neq d_2. \end{cases}$$

Corollary

Let E and F be finite graphs.

$$K_0^{gr}(L_K(E)) \cong K_0^{gr}(L_K(F)) \implies GKdim\ L_K(E) = GKdim\ L_K(F).$$

$$\begin{tikzcd}
 & E \arrow[dl] \arrow[dr] & \\
L_K(E) \arrow[rr, dash, dashed] & & T_E
\end{tikzcd}$$

$$
\begin{tikzcd}
 & E \arrow[dl] \arrow[dr] & \\
L_K(E) \arrow[rr, dashed, no head] & & T_E \\
{[L_K(E), L_K(E)]} \arrow[u] & &
\end{tikzcd}
$$

Lie Bracket

Lie Algebra

A *Lie algebra* over a field K is a K-vector space L together with a map

$$[-,-] : L \times L \longrightarrow L$$

such that

(1) $[-,-]$ is bilinear,
(2) $[x,x] = 0$ for all $x \in L$,
(3) $[x,[y,z]] + [y,[z,x]] + [[z,[x,y]]] = 0$ for all $x,y,z \in L$.

Let K be a field and A an associative K-algebra.

$$[x,y] := xy - yx$$

$$\Downarrow$$

$(A, [-,-])$ – associated Lie algebra

Lie Solvable and Lie Nilpotent

Lie Solvable and Lie Nilpotent

Let $(L, [-,-])$ be a Lie algebra. Define

$$\begin{aligned} L &= L^{(0)} \\ &= L^{0} \\ L^{(n)} &= [L^{(n-1)}, L^{(n-1)}] \\ L^{n} &= [L, L^{n-1}] \end{aligned}$$

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The associative algebra A is called *Lie solvable* (resp. *Lie nilpotent*) of index n if its associated Lie algebra is solvable (resp. nilpotent) of index n.

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(i) $\mathrm{char}(K) = 2$
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 (b) or a cyclic ideal with $|\langle v \rangle \cap E^0| \leq 2$,
 (c) or for every $e \in s^{-1}(v)$, $r(e) \parallel w$ as elements of M_E for every $w \neq v$.

(ii) $\mathrm{char}(K) \neq 2$
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Simplicity

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We call a vertex v in a connected graph E a *balloon* over a nonempty set $W \subseteq E^0$ if

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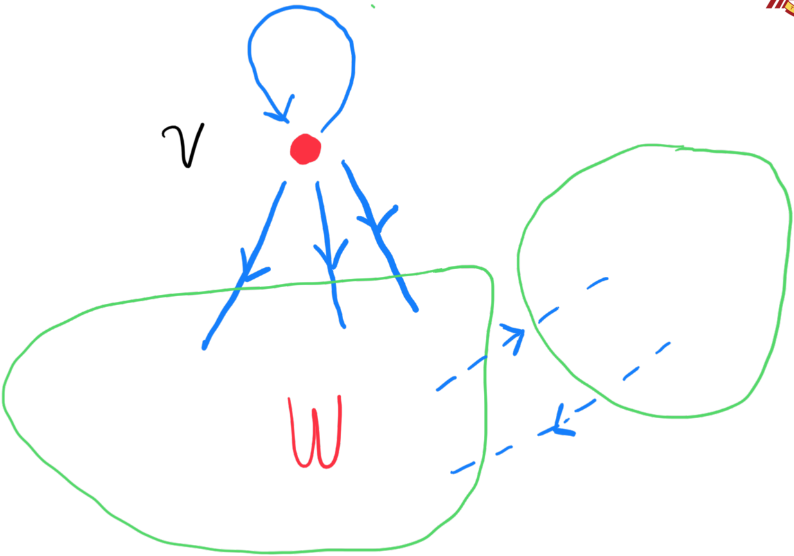
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(iii) $T_{E/H}$ is simple cyclic.

Theorem

Let E be connected row-finite graph with $L_K(E)$ not simple. $[L_K(E), L_K(E)]$ is simple $\Leftrightarrow$ for every vertex $v \notin I$ for some $\mathbb{Z}$-order-ideal I, Theorem ❤ (i)-(iii) are satisfied and

$$\sum_{w \in r(E(v,W))} w \in [L_K(W), L_K(W)]$$

where $W = E^o \cap J$, J the minimal non-cyclic $\mathbb{Z}$-order-ideal of T_E.

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$$1_{L_K(E)} = \sum_{v \in E^0} v \notin [L_K(E), L_K(E)].$$

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