Bicategories, 2-monads, enriched and internal categories

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(minicourse, advanced)

Cimpa School
"From Dynamics to Algebra and Rep. Theory and Back"

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Mathematical Institute of
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Overview of the 2nd class

- categories and monoidal categories
- bicategories and 2-categories
- strictification theorems

- double categories (as specific internal categories)
- monads and 2-monads

- internal categories (as specific 2-monads)
- enriched categories and when they induce internal categories
- 2-monads in the bicategories of spans and matrices
- double category of monads (why "vertical morphisms" of monads are useful)

Internal categories

Recall: the bicategory Span($\mathcal{C}$) of spans in $\mathcal{C}$ with pullbacks

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2-cells: commutative diagrams in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[d, "="'] & X \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "f"] & B \arrow[d, "="] \\ A & X' \arrow[l, "a'"] \arrow[r, "b'"'] & B \end{tikzcd}$$

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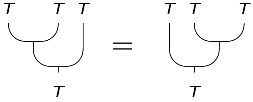
Recall that a 2-monad consists of:

- a 0-cell $\mathcal{A}$,
- a 1-cell $T : \mathcal{A} \longrightarrow \mathcal{A}$,
- 2-cells $\mu : TT \Rightarrow T$ and $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$
- s.t. axioms hold:



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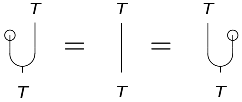

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- 2-cells $\mu : TT \Rightarrow T$ and $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$ (commutative diagrams in $\mathcal{C}$)

for 2-cell μ:

$$\begin{tikzcd} X_0 \arrow[d, "="'] & X_1 \times_{X_0} X_1 \arrow[l, "tp_1"'] \arrow[r, "sp_2"] \arrow[d, "c"] & X_0 \arrow[d, "="] \\ X_0 & X_1 \arrow[l, "t"] \arrow[r, "s"'] & X_0 \end{tikzcd}$$

for 2-cell η:

$$\begin{tikzcd} X_0 \arrow[d, "="'] & X_0 \arrow[l, "id"'] \arrow[r, "id"] \arrow[d, "u"] & X_0 \arrow[d, "="] \\ X_0 & X_1 \arrow[l, "t"] \arrow[r, "s"'] & X_0 \end{tikzcd}$$

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s.t. $c : X_1 \times_{X_0} X_1 \longrightarrow X_1$ is associative and unital with respect to $u : X_0 \longrightarrow X_1$

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and which satisfy: $tc = tp_1, sc = sp_2, tu = id_{X_0} = su$.

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- A category internal in Top: a topological category.

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- A cat. enriched over $PsDbl_2$ $=$ *locally cubical bicategory* [Garner-Gurski].

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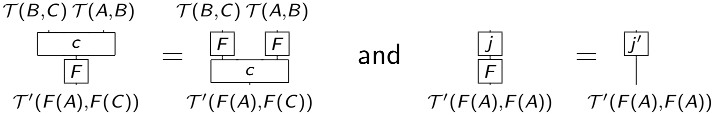

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When enriched cats are internal cats

[Ehresmann & Ehresmann (1978)]

Proposition.

Assume:

- $\mathcal{C}$ a category with finite products, a terminal object $\mathcal{I}$ and small coproducts.
- $\mathcal{T}$ is a category enriched over $\mathcal{C}$, set $T_0 = \amalg_{A \in \mathcal{O}b\mathcal{T}} \mathcal{I}_A$ and $T_1 = \amalg_{A,B \in \mathcal{O}b\mathcal{T}} \mathcal{T}(A,B)$, $\mathcal{C}$ has iterated pullbacks $T_1^{(n)_0}$;
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• There is a functor $\Gamma : \mathcal{C}\text{- Cat} \longrightarrow \mathrm{Cat}(\mathcal{C})$ which corestricts to an equivalence $\Gamma' : \mathcal{C}\text{- Cat} \longrightarrow \mathrm{Cat}_d(\mathcal{C})$.

The bicategory of matrices
$\mathcal{C}$- Mat

($\mathcal{C}$ a category with products and coproducts)

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Identity 2-cell: identity morphism on $(M(i,j))_{i \in I, j \in J}$

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Given 2-cells:
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<u>Vertical composition of 2-cells:</u> $M(i,j) \xrightarrow{f_{i,j}} N(i,j) \xrightarrow{g_{i,j}} L(i,j)$

Enriched categories as 2-monads in $\mathcal{C}\text{-}\mathsf{Mat}$

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...are categories enriched over $\mathcal{C}$.

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we conclude: a 2-monad in the bicat. $\mathcal{C}$- Mat is a **cat. enriched** / $\mathcal{C}$.

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Theorem. [Cottrell, Fujii, Power (2017)] Let V be a cartesian closed category with finite limits and small coproducts. TFAE:

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The 2-category $\mathsf{Mnd}(\mathcal{K})$

(of 2-monads)

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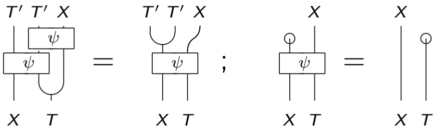

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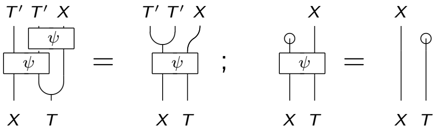
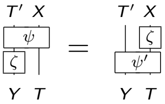


<u>2-cells:</u> $(X, \psi) \Rightarrow (Y, \psi')$ are given by 2-cells $\zeta : X \longrightarrow Y$ in $\mathcal{K}$ satisfying:



Compositions in Mnd($\mathcal{K}$)

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Why 1-cells in $\mathsf{Mnd}(\mathcal{C}\text{-}\mathsf{Mat})$

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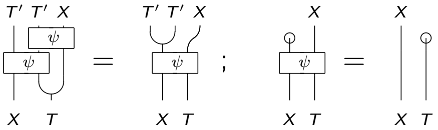

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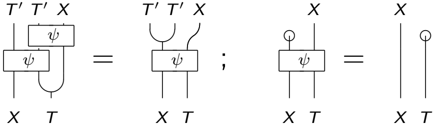

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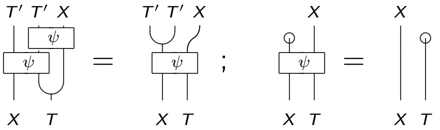

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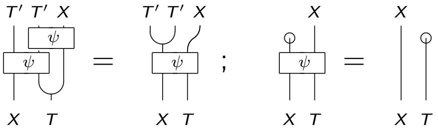

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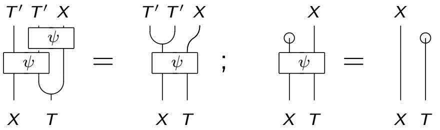

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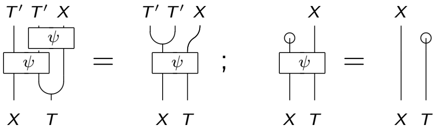

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in $\mathcal{C}$- Mat, satisfying above laws.

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Recall: a 2-monad in the bicat. $\mathcal{C}$- Mat is a cat. enriched / $\mathcal{C}$ by setting:
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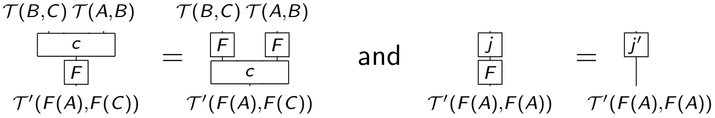
A functor $F : \mathcal{T} \longrightarrow \mathcal{T}'$, with $(\mathcal{T}, c, j)$ and $(\mathcal{T}', c', j')$, is $\mathcal{C}$-enriched if the maps $F_{A,B} : \mathcal{T}(A,B) \longrightarrow \mathcal{T}'(F(A), F(B))$ are morphisms in $\mathcal{C}$, s.t.



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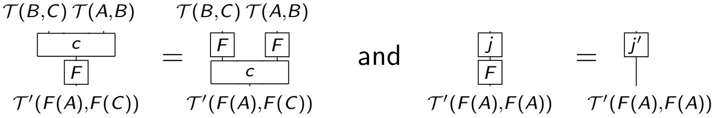

$\Rightarrow$ <u>1-cells in Mnd($\mathcal{C}$- Mat) are not $\mathcal{C}$-enriched functors</u>

The double category of 2-monads

Solution: the double category Mnd($\mathbb{D}$) of (double) monads

[Fiore, Gambino, Kock (2010)]

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This ... allows us to **describe mathematical structures and morphisms between them as monads and vertical monad maps in appropriate double categories**. E.g. small categories and functors can be viewed as monads and vertical monad maps in the the double category of spans.

The double category Mnd($\mathbb{D}$) of (double) monads

0-cells:

Definition 2.4. Let $\mathbb{C}$ be a double category.

(i) A *monad* is an endomorphism (X, P) equipped with squares

$$\begin{tikzcd} X \arrow[r, "P"] \arrow[d, equal] \arrow[drr, phantom, "\mu_P"] & X \arrow[r, "P"] & X \arrow[d, equal] \\ X \arrow[rr, "P"'] & & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta_P"] & X \arrow[d, equal] \\ X \arrow[r, "P"'] & X \end{tikzcd}$$

satisfying the associativity law

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] \arrow[drr, phantom, "\mu_P"] & X \arrow[r] & X \arrow[r] \arrow[d, equal] & X \arrow[d, equal] \\ X \arrow[rr] \arrow[d, equal] \arrow[drrr, phantom, "\mu_P"] & & X \arrow[r] & X \arrow[d, equal] \\ X \arrow[rrr] & & & X \end{tikzcd} = \begin{tikzcd} X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, equal] \arrow[drr, phantom, "\mu_P"] & X \arrow[r] & X \arrow[d, equal] \\ X \arrow[r] \arrow[d, equal] \arrow[drrr, phantom, "\mu_P"] & X \arrow[rr] & & X \arrow[d, equal] \\ X \arrow[rrr] & & & X \end{tikzcd}$$

and the unit laws

.....

The double category Mnd($\mathbb{D}$) of (double) monads

1v-cells:

(iii) A *vertical monad map* $(u, \bar{u}) : (X, P) \to (X', P')$ is a vertical endomorphism map between the underlying endomorphisms satisfying the following conditions:

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, phantom, "\mu_P"] & X \arrow[d, equal] \\
X \arrow[rr] \arrow[d] & \arrow[d, phantom, "\bar{u}"] & X \arrow[d] \\
X' \arrow[rr] & & X'
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\bar{u}"] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\bar{u}"] & X \arrow[d] \\
X' \arrow[r] \arrow[d, equal] & X' \arrow[r] \arrow[d, phantom, "\mu_{P'}"] & X' \arrow[d, equal] \\
X' \arrow[rr] & & X'
\end{tikzcd}
$$

$$
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=
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X' \arrow[r] & X'.
\end{tikzcd}
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$$
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\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X \arrow[d] \\
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$\Rightarrow$ 1v-cells in Mnd($\underline{\underline{\mathcal{C}\text{- Mat}}}$) are $\mathcal{C}$-enriched functors.

Recovering Ehresmann's result

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We saw that:
the oplax functor $\mathsf{Int} : \mathcal{C}\text{-}\mathsf{Mat} \longrightarrow \mathsf{Span}_d(\mathcal{C})$ (biequivalence)
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And one obtains:

Theorem. Let V be a cartesian closed category with finite limits and small coproducts. TFAE: 1. V is *extensive*, and 2. The **double functor** $\mathit{Int} : \mathcal{C}\text{-}\underline{\underline{\mathrm{Mat}}} \longrightarrow \underline{\underline{\mathrm{Span}}}_d(\mathcal{C})$ **is a double equivalence**.

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Corollary. Let $\mathcal{C}$ be an *extensive* cartesian closed category with finite limits and small coproducts. Then Γ' : $\mathcal{C}$- Cat $\longrightarrow \mathrm{Cat}_d(\mathcal{C})$ is an adjoint equivalence functor.