## Bicategories, <br> 2-monads, enriched and internal categories

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(minicourse, advanced)
Cimpa School
"From Dynamics to Algebra and Rep. Theory and Back"

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Mathematical Institute of
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Belgrade (Serbia)

## Overview of the 2 nd class

- categories and monoidal categories
- bicategories and 2-categories
- strictification theorems
- double categories (as specific internal categories)
- monads and 2-monads
- internal categories (as specific 2-monads)
- enriched categories and when they induce internal categories
- 2-monads in the bicategories of spans and matrices
- double category of monads (why "vertical morphisms" of monads are useful)


## Double categories

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Bicategory:

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- horizontal 1-cells
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$C_{0}$ : 0 -cells and 1 v -cells, $C_{1}: 1 \mathrm{~h}$-cells and 2 -cells.


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$s, t: C_{1} \rightarrow C_{0}, \quad u: C_{0} \rightarrow C_{1} \quad$ and $c: C_{1} \times c_{0} C_{1} \rightarrow C_{1}$
- natural transformations (2-cells in $\mathrm{Cat}_{2}$ )
$\alpha: c \otimes\left(i d C_{C_{1}} \times c_{0} c\right) \Rightarrow c \otimes\left(c \times c_{0} i d_{C_{1}}\right)$
$\lambda: c \otimes\left(u \times C_{0} i d_{C_{1}}\right) \Rightarrow i d_{C_{1}}$ $\rho: c \otimes\left(i d_{C_{1}} \times C_{0} u\right) \Rightarrow i d_{C_{1}}$
which satisfy a pentagon and a triangle.


## Bicategories and pseudodouble categories

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$\alpha: M \rightarrow N \quad A$ - $B$-bimodule morphism

$$
a \cdot n \cdot b:=g(a) \cdot n \cdot f(b)
$$

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$$
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- Recall that every pseudodouble category is double-equivalent to a double category.


## Monads

## and 2-monads

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Morphisms of monads
are nat. transf. $f: T \rightarrow P$ s.t.:


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- A monad on a category $\mathcal{C}(T: \mathcal{C} \rightarrow \mathcal{C})$ is a 2 -monad in $\mathcal{K}=\mathrm{Cat}_{2}$ (with $\mathcal{A}=\mathcal{C}$ ).


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Cat $_{1}$ as a category is a 0 -cell in $\mathrm{Cat}_{2}$,
$T$ as a functor is a 1 -cell in $\mathrm{Cat}_{2}$,
and $\mu_{T}, \eta_{T}$ as natural transformations are 2-cells in Cat ${ }_{2}$.

## 2-monads yielding strict monoidal categories

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$$
(T T(*)=T(\mathcal{C}) \stackrel{\text { cond. }}{=} \mathcal{C} \times \mathcal{C})
$$

## The 2-category $\operatorname{Mnd}(\mathcal{K})$

(of 2-monads)

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0 -cells:
2-monads $\left(\mathcal{A}, T: \mathcal{A} \rightarrow \mathcal{A}, \mu_{T}: T T \rightarrow T, \eta_{T}: \operatorname{ld}_{\mathcal{A}} \rightarrow T\right)$

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$$
\begin{aligned}
& \underset{\left.\right|_{T} ^{\psi}}{\substack{x}}=\left.\left.\right|_{x T} ^{x}\right|_{T} ^{x}
\end{aligned}
$$

## The 2-category $\operatorname{Mnd}(\mathcal{K})$ of monads in $\mathcal{K}$

0 -cells:
2-monads $\left(\mathcal{A}, T: \mathcal{A} \rightarrow \mathcal{A}, \mu_{T}: T T \rightarrow T, \eta_{T}: \operatorname{ld}_{\mathcal{A}} \rightarrow T\right)$
1-cells: pairs $(X, \psi):(\mathcal{A}, T) \rightarrow\left(\mathcal{A}^{\prime}, T^{\prime}\right)$ where $X: \mathcal{A} \rightarrow \mathcal{A}^{\prime}$ is a 1-cell and $\psi: T^{\prime} X \Rightarrow X T$ a 2 -cell s.t.


2-cells: $(X, \psi) \Rightarrow\left(Y, \psi^{\prime}\right)$ are given by 2-cells $\zeta: X \rightarrow Y$ in $\mathcal{K}$ satisfying:

$$
\begin{gathered}
T^{\prime} x \\
\Psi_{\psi}^{\psi} \\
\hline \zeta \\
Y T
\end{gathered}=\begin{gathered}
T^{\prime} x \\
Y \zeta \\
Y \psi^{\prime} \\
Y T
\end{gathered}
$$

Compositions in $\mathrm{Mnd}(\mathcal{K})$

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Set $X(*)=\mathcal{M}$. Then $X=\mathcal{M} \times-$ is a functor (1-cell in Cat ${ }_{2}$ ), and $\nu: \mathcal{C} \times(\mathcal{M} \times-) \Rightarrow \mathcal{M} \times-$ a natural transformation
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Conversely: $\mathcal{M}$ yields a $T$-algebra $X$ with $T=\mathcal{C} \times-, X=\mathcal{M} \times-$.


## 2-Monads as lax functors

## Lax functors

A lax functor $\mathcal{F}: \mathcal{B} \rightarrow \mathcal{B}^{\prime}$

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\mathcal{F}\left(\frac{\alpha}{\beta}\right)=\frac{\mathcal{F}(\alpha)}{\mathcal{F}(\beta)}
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- 2-cells natural in $g, f$ :
$\Rightarrow \mathcal{F}(g) \circ \mathcal{F}(f) \xrightarrow{\xi} \mathcal{F}(g \circ f)$
+ hexagon;
$-i d_{\mathcal{F}(*)} \xrightarrow{\zeta} \mathcal{F}(i d)$
+2 squares.

Lax functors $* \rightarrow \mathcal{K}$

A lax functor $\mathcal{F}: * \mathcal{K}$

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objects (1-cells):
id $\mapsto \mathcal{F}(i d):=T$
morphisms (2-cells):

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\mathrm{Id}_{i d} \mapsto \mathcal{F}\left(\mathrm{Id}_{i d}\right):=\mathrm{Id}_{T}
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$$

morphisms (2-cells): $\quad \quad \operatorname{Id}_{i d} \mapsto \mathcal{F}\left(\mathrm{Id}_{i d}\right):=\mathrm{Id}_{T}$

- 2-cells (natural in) id:
$\triangleright \mathcal{F}(i d) \circ \mathcal{F}(i d) \xrightarrow{\xi} \mathcal{F}(i d \circ i d) \quad:=\mu_{T} \quad+$ hexagon;
$\triangleright i d_{\mathcal{F}(*)} \xrightarrow{\zeta} \mathcal{F}\left(i d_{*}\right) \quad:=\eta_{T}$
+2 squares.


## Diagrams for pseudofunctors

$$
\begin{aligned}
& \mathcal{F}(h) \circ(\mathcal{F}(g) \circ \mathcal{F}(f)) \xrightarrow{1 \circ \xi} \mathcal{F}(h) \circ \mathcal{F}(g \circ f) \xrightarrow{\mathcal{F}}(h \circ(g \circ f))
\end{aligned}
$$

$$
\begin{aligned}
& \underset{\rho}{\left.\mathcal{F}(f) \circ \mathrm{Id}_{A} \xrightarrow{1 \circ \zeta} \mathcal{F}(f) \circ \mathcal{F}\left(\mathrm{Id}_{A}\right) \xrightarrow[\mathcal{F}(\rho)]{\underset{\mathcal{F}}{\boldsymbol{G}} \mathcal{F}}\left(f \circ \mathrm{Id}_{A}\right)\right)}
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$$

$$
\begin{aligned}
& \mathrm{Id}_{B} \circ \mathcal{F}(f) \underset{\mathcal{F}(f)}{\zeta \circ 1} \mathcal{F}\left(\mathrm{Id}_{B}\right) \circ \mathcal{F}(f) \xrightarrow{\underline{\mathcal{F}}(\lambda)} \mathcal{F}\left(\mathrm{Id}_{B} \circ f\right) \\
& \underset{\rho}{\mathcal{F}(f) \circ \mathrm{Id}_{A} \xrightarrow{1 \circ \zeta} \mathcal{F}(f) \circ \mathcal{F}\left(\mathrm{Id}_{A}\right) \xrightarrow{\boldsymbol{F}(f)} \underset{\mathcal{F}(\rho)}{\mathcal{F}}\left(f \circ \mathrm{Id}_{A}\right)}
\end{aligned}
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$\Rightarrow$ A lax functor $\mathcal{F}: * \rightarrow \mathcal{K}$ is a 2 -monad in $\mathcal{K}$.

