Bicategories, 2-monads, enriched and internal categories

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(minicourse, advanced)

Cimpa School
"From Dynamics to Algebra and Rep. Theory and Back"

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Mathematical Institute of
Serbian Academy of Sciences and Arts
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Overview of the course

- categories and monoidal categories
- bicategories and 2-categories
- strictification theorems

- double categories (as specific internal categories)
- monads and 2-monads

- internal categories (as specific 2-monads)
- enriched categories and when they induce internal categories
- 2-monads in the bicategories of spans and matrices
- double category of monads (why "vertical morphisms" of monads are useful)

Categories

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s.t. $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ and $\mathcal{F}(id_A) = id_{\mathcal{F}(A)}$.

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s.t. for all $f : A \longrightarrow B$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\omega(A)"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(A) \arrow[d, "\mathcal{G}(f)"] \\
\mathcal{F}(B) \arrow[r, "\omega(B)"] & \mathcal{G}(B)
\end{tikzcd}$$

Monoidal categories

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- functors $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and $U : * \longrightarrow \mathcal{C}$ (the unit $I \in \mathcal{O}b\mathcal{C}$)

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so that the following two diagrams commute:

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes W \arrow[r, "\alpha_{X,Y,Z} \otimes W"] \arrow[d, "\alpha_{X \otimes Y,Z,W}"'] & (X \otimes (Y \otimes Z)) \otimes W \arrow[r, "\alpha_{X,Y \otimes Z,W}"] & X \otimes ((Y \otimes Z) \otimes W) \arrow[d, "X \otimes \alpha_{Y,Z,W}"] \\
(X \otimes Y) \otimes (Z \otimes W) \arrow[rr, "\alpha_{X,Y,Z \otimes W}"'] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[dr, "\rho_X \otimes Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "X \otimes \lambda_Y"] \\
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Examples of monoidal categories

The category of vector spaces over a filed k: $(Vec_k, \otimes_k, k)$.

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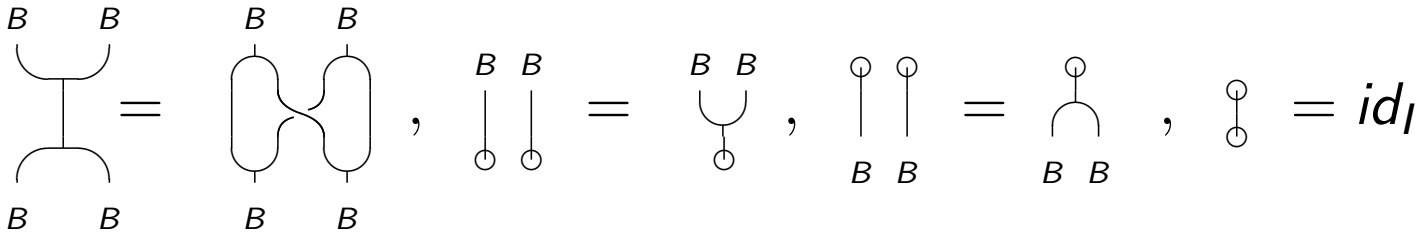
Moreover, if B is a (quasi)bialgebra in a braided monoidal category $(\mathcal{C}, \otimes, I, \Phi)$:



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Bicategories and 2-categories

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for $A, B, C \in \mathcal{O}b\mathcal{B}$,

associative, compatible with Id **up to isomorphisms** α, λ, ρ,
(natural transformations)
that satisfy pentagon + triangle.

Bicategories

(5) *for each quadruple of objects* $A, B, C, D \in \mathfrak{A}$ *an isomorphism, natural in* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ *called the* associativity law

$$\alpha_{h,g,f} : h \circ (g \circ f) \Rightarrow (h \circ g) \circ f.$$

and for each pair of objects $A, B \in \mathfrak{A}$ *two in* $f : A \to B$ *natural isomorphisms, called the* left- *and the* right unitary law, *respectively.*

$$\lambda_f : \mathrm{Id}_B \circ f \Rightarrow f \qquad \rho_f : f \circ \mathrm{Id}_A \Rightarrow f$$

such that the following diagrams commute for all $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$

$$\begin{tikzcd}
k \circ (h \circ (g \circ f)) \arrow[r, "1 \circ \alpha"] \arrow[d, "\alpha"'] & k \circ ((h \circ g) \circ f) \arrow[r, "\alpha"] & (k \circ (h \circ g)) \circ f \arrow[d, "\alpha \circ 1"] \\
(k \circ h) \circ (g \circ f) \arrow[rr, "\alpha"] & & ((k \circ h) \circ g) \circ f
\end{tikzcd}$$

$$\begin{tikzcd}
g \circ (\mathrm{Id}_B \circ f) \arrow[r, "\alpha"] \arrow[dr, "1 \circ \lambda"'] & (g \circ \mathrm{Id}_B) \circ f \arrow[d, "\rho \circ 1"] \\
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Tricategory:

0: tensor cats $\mathcal{C}, \mathcal{D}$...,
1: $\mathcal{C}$-$\mathcal{D}$-bimodule cats,
2: $\mathcal{C}$-$\mathcal{D}$-bimodule-functors,
3: $\mathcal{C}$-$\mathcal{D}$-bimodule nat. transf.'s

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In **tricategories** there is a **3-cell** controlling the interchange law:

$$\xi : \frac{(\alpha|\beta)}{(\gamma|\delta)} \xrightarrow{\cong} \left(\frac{\alpha}{\gamma}\middle|\frac{\beta}{\delta}\right).$$

The bicategory of spans $\mathbf{Span}(\mathcal{C})$

($\mathcal{C}$ a category with pullbacks)

Recall: pullback in $\mathcal{C}$

A **pullback** over an object B with respect to morphisms $f : X \longrightarrow B$ and $g : Y \longrightarrow B$ in $\mathcal{C}$ is given by:

1. an object P (often written as $X \times_B Y$) and morphisms $p_1 : P \longrightarrow X, p_2 : P \longrightarrow Y$ so that $fp_1 = gp_2$

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

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2. for every object T and morphisms $q_1 : T \longrightarrow X, q_2 : T \longrightarrow Y$ such that $fq_1 = gq_2$ there is a unique morphism $t : T \longrightarrow P$, such that $tp_1 = q_1$ and $tp_2 = q_2$.

$$\begin{tikzcd}
T \arrow[dr, red, "t"] \arrow[drr, "q_2"] \arrow[ddr, "q_1"'] & & \\
& P \arrow[r, "p_2"'] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
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- Pullbacks are unique up to unique isomorphisms.

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2-cells: commutative diagrams in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[d, "="'] & X \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "f"] & B \arrow[d, "="] \\ A & X' \arrow[l, "a'"'] \arrow[r, "b'"] & B \end{tikzcd}$$

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vertical composition of 2-cells:

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The bicategory of spans in $\mathcal{C}$ with pullbacks

horizontal composition of 1- and 2-cells:

$$
\begin{tikzcd}
 & & X \times_B Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "h"] & & \\
 & X \arrow[dl, "a"'] \arrow[dr, "b_2"] \arrow[dd, "f"] & & Y \arrow[dl, "b_1"'] \arrow[dr, "c"] \arrow[dd, "g"] & \\
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The bicategory of spans in $\mathcal{C}$ with pullbacks

horizontal composition of 1- and 2-cells:

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Associativity and (left and right) unity natural isomorphisms α, λ, ρ (2-cells) are given by the usual respective isomorphisms of pullbacks.

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- for all composable 1-cells g, f natural isomorphism 2-cells:
  - $\mathcal{F}(g \circ f) \overset{\xi}{\cong} \mathcal{F}(g) \circ \mathcal{F}(f)$ + hexagon;
  - $\mathcal{F}(id_A) \overset{\zeta}{\cong} id_{\mathcal{F}(A)}$ + 2 squares.

Pseudofunctors

$$\begin{tikzcd}
\mathcal{F}(h) \circ (\mathcal{F}(g) \circ \mathcal{F}(f)) \arrow[r, "1 \circ \xi"] \arrow[d, "\alpha"] & \mathcal{F}(h) \circ \mathcal{F}(g \circ f) \arrow[r, "\xi"] & \mathcal{F}(h \circ (g \circ f)) \arrow[d, "\mathcal{F}(\alpha)"] \\
(\mathcal{F}(h) \circ \mathcal{F}(g)) \circ \mathcal{F}(f) \arrow[r, "\xi \circ 1"] & \mathcal{F}(h \circ g) \circ \mathcal{F}(f) \arrow[r, "\xi"] & \mathcal{F}((h \circ g) \circ f)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Id}_B \circ \mathcal{F}(f) \arrow[r, "\zeta \circ 1"] \arrow[dr, "\lambda"'] & \mathcal{F}(\mathrm{Id}_B) \circ \mathcal{F}(f) \arrow[r, "\xi"] & \mathcal{F}(\mathrm{Id}_B \circ f) \arrow[dl, "\mathcal{F}(\lambda)"] \\
& \mathcal{F}(f) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(f) \circ \mathrm{Id}_A \arrow[r, "1 \circ \zeta"] \arrow[dr, "\rho"'] & \mathcal{F}(f) \circ \mathcal{F}(\mathrm{Id}_A) \arrow[r, "\xi"] & \mathcal{F}(f \circ \mathrm{Id}_A) \arrow[dl, "\mathcal{F}(\rho)"] \\
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A natural isomorphism ξ for the composition of 1-cells:

$$\mathcal{M} \boxtimes_{\mathcal{D}} (\mathcal{F} \circ \mathcal{G}) \longrightarrow (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{F}) \circ (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{G}).$$

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\mathcal{F}(A) \arrow[r, "\chi_A"] \arrow[d, "\mathcal{F}(f)"'] \arrow[dr, phantom, "\boxed{\chi_f}"] & \mathcal{G}(A) \arrow[d, "\mathcal{F}(g)"] \\
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s.t.

$$
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\cdot \arrow[r, "\mathcal{F}(f)"] \arrow[d, "="'] & \cdot \arrow[r, "\chi_B"] \arrow[d, "="] \arrow[dr, phantom, "\boxed{\chi_f}"] & \cdot \arrow[r, "\mathcal{G}(g)"] \arrow[d, "="] & \cdot \\
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$$

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\cdot \arrow[rr, "\chi_A"] \arrow[d, "="] & & \cdot \arrow[d, "="] \\
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When α, λ, ρ in a monoidal category are identities := **strict monoidal category**. (Example: $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$)

When α, λ, ρ in a bicategory are identities := **2-category**.

A tricategory in which only the **interchange law** is an isomorphism (the rest is strict) := **Gray category**.

Strictification Theorems:

- Each monoidal cat. is equivalent to a strict monoidal cat.
- Each bicategory is bi-equivalent to a 2-category.
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- (For what comes next:) Each pseudodouble category is double-equivalent to a double category.